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Cooling Process for Inelastic Boltzmann Equations for Hard Spheres, Part II: Self-Similar Solutions and Tail Behavior

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We consider the spatially homogeneous Boltzmann equation for inelastic hard spheres, in the framework of so-called *constant normal restitution coefficients*. We prove the existence of self-similar solutions, and we give pointwise estimates on their tail. We also give general estimates on the tail and the regularity of generic solutions. In particular we prove Haff's law on the rate of decay of temperature, as well as the algebraic decay of singularities. The proofs are based on the regularity study of a rescaled problem, with the help of the regularity properties of the gain part of the Boltzmann collision integral, well-known in the elastic case, and which are extended here in the context of granular gases.

KEY WORDS: Boltzmann equation, inelastic hard spheres, granular gas, cooling process, Haff's law, self-similar solutions, regularity of the collision operator, tail behavior.
Mathematics Subject Classification (2000): 76P05 Rarefied gas flows, Boltzmann equation [See also 82B40, 82C40, 82D05].

1. INTRODUCTION AND MAIN RESULTS

1.1. The Model

We consider the asymptotic behavior of inelastic hard spheres described by the spatially homogeneous Boltzmann equation with a *constant normal restitution coefficient* (see Ref. 28). More precisely, the gas is described by the probability density of particles $f(t, v) \geq 0$ with velocity $v \in \mathbb{R}^N$ $(N \geq 2)$ at time $t \geq 0$, which

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undergoes the evolution equation

$$\frac{\partial f}{\partial t} = Q(f, f) \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^N, \tag{1.1}$$

$$f(0) = f_{\text{in}} \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

The bilinear collision operator $Q(f, f)$ models the interaction of particles by means of inelastic binary collisions (preserving mass and momentum but dissipating kinetic energy). Denoting by $e \in [0, 1]$ the (constant) *normal restitution coefficient*, when $e \neq 0$ we define the collision operator in strong formulation as

$$Q(g, f)(v) := \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left(\frac{{}'f\, {}'g_*}{e^2} - f g_* \right) |u|\, b(\hat{u} \cdot \sigma)\, d\sigma\, dv_*, \tag{1.3}$$

where we use notations from Ref. 19 (a dual formulation shall be given in (1.25) which includes the case $e = 0$). Here $u = v - v_*$ denotes the relative velocity, $\hat{u}$ stands for $u/|u|$, and ${}'v$, ${}'v_*$ denotes the possible pre-collisional velocities leading to post-collisional velocities v, v_*. They are defined by

$${}'v = \frac{v + v_*}{2} + \frac{{}'u}{2}, \qquad {}'v_* = \frac{v + v_*}{2} - \frac{{}'u}{2}, \tag{1.4}$$

with ${}'u = (1 - \beta)u + \beta |u| \sigma$ and $\beta = (e + 1)/(2e)$ ($\beta \in [1, \infty)$ since $e \in (0, 1]$). The elastic case corresponds to $e = 1$. The function b in (1.3) is (up to a multiplicative factor) the differential collisional cross-section while $B = |u|\, b(\hat{u} \cdot \sigma)$ represents the rate of collision of particles with pre-collisional velocities $v, v_* \in \mathbb{R}^N$ giving rise to particles with post-collisional velocities $v', v'_* \in \mathbb{R}^N$ defined by (1.26). In the sequel we assume that there exists $b_0, b_1 \in (0, \infty)$ such that

$$\forall x \in [-1, 1], \quad b_0 \le b(x) \le b_1, \tag{1.5}$$

and that

$$b \text{ is nondecreasing and convex on } (-1, 1). \tag{1.6}$$

Note that the "physical" cross-section for hard spheres is given by (see Refs. 14, 19)

$$b(x) = \text{cst}\,(1 - x)^{-\frac{N-3}{2}},$$

so that it fulfills hypothesis (1.5) and (1.6) when $N = 3$. The Boltzmann equation (1.1) is complemented with an initial datum (1.2) which satisfies (for some $k \ge 2$)

$$0 \le f_{\text{in}} \in L^1_k(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} f_{\text{in}}\, dv = 1, \qquad \int_{\mathbb{R}^N} f_{\text{in}}\, v\, dv = 0. \tag{1.7}$$

(see Sec. 1.4 for the notations of functional spaces). Notice that assuming the two last moment conditions in (1.7) is no loss of generality, since we may always reduce to that case by a scaling and translation argument (see [19, Sec. 1.5] for instance).

As explained in Ref. 28, the operator (1.3) preserves mass and momentum:

$$\frac{d}{dt}\int_{\mathbb{R}^N} f \begin{pmatrix} 1 \\ v \end{pmatrix} dv = 0, \tag{1.8}$$

while kinetic energy is dissipated

$$\frac{d}{dt}\mathcal{E}(f(t,\cdot)) = -D(f(t,\cdot)) \tag{1.9}$$

where the energy $\mathcal{E}$ and the dissipation functional D are given by

$$\mathcal{E}(f) = \int_{\mathbb{R}^N} f(v)\,|v|^2\,dv, \qquad D(f) := \tau \int_{\mathbb{R}^N\times\mathbb{R}^N} f\,f_*\,|u|^3\,dv\,dv_*.$$

Here the inelasticity coefficient τ is defined by $\tau := m_b\,(\frac{1-e^2}{4})$ and the angular momentum m_b is defined by

$$m_b := \int_{\mathbb{S}^{N-1}} \left(\frac{1-(\hat{u}\cdot\sigma)}{2}\right) b(\hat{u}\cdot\sigma)\,d\sigma = |\mathbb{S}^{N-2}| \int_0^{\pi} b(\cos\theta)\,\sin^2\theta/2\,\sin^{N-2}\theta\,d\theta$$

(in order to get the second formula, we have set $\cos\theta = \hat{u}\cdot\sigma$).

The study of the Cauchy theory and the cooling process of (1.1)–(1.2) was done in Ref. 28 (where more general models were considered). The equation is well-posed for instance in L^1_2: for $0 \le f_{\text{in}} \in L^1_2$, there is a unique solution in $C(\mathbb{R}_+; L^1_2) \cap L^1(\mathbb{R}_+; L^1_3)$ (see again Sec. 1.4 for the notations of functional spaces). This solution is defined for all times. It preserves mass, momentum and has a decreasing kinetic energy. The cooling process does not occur in finite time, but asymptotically in large time, *i.e.*, the kinetic energy is strictly positive for all times and the solution satisfies

$$\mathcal{E}(t) \to 0 \quad \text{and} \quad f(t,\cdot) \rightharpoonup \delta_{v=0} \quad \text{in} \quad M^1(\mathbb{R}^N)\text{-weak}\,* \quad \text{when} \quad t \to +\infty,$$

where $M^1(\mathbb{R}^N)$ denotes the space of probability measures on $\mathbb{R}^N$. We refer to Ref. 28 for the proofs of these results.

1.2. Introduction of Rescaled Variables

Let us introduce some rescaled variables, in order to study more precisely the asymptotic behavior of the solution. This usual rescaling can be found in Refs. 9 and 16 for instance. We search for a rescaled solution g of the form

$$f(t,v) = K(t)\,g(T(t), V(t)v), \tag{1.10}$$

where K, T, V are time scaling functions to be determined, such that $K(0) = V(0) = 1$ and $T(0) = 0$ (same initial datum). We choose the scaling functions K, V such that they are compatible with *self-similar solutions*, that is when g does not depend on time: there exists a profile function G such that

$$f(t, v) = K(t)\, G(V(t)v). \tag{1.11}$$

In this case, the conservation of mass

$$\mathrm{cst} = \int_{\mathbb{R}^N} f(t, v)\, dv = \frac{K(t)}{V(t)^N} \int_{\mathbb{R}^N} G(w)\, dw$$

implies $K(t) = V(t)^N$. The evolution Eq. (1.1) satisfied by f implies therefore

$$V'(t)\, \nabla_v \cdot (v\, G) = Q(G, G) \tag{1.12}$$

by using the following homogeneity property: for any function g such that the collision operator is well-defined,

$$\forall\, \lambda \in \mathbb{R}^*, \quad Q(g(\lambda \cdot), g(\lambda \cdot))(v) = \lambda^{-(N+1)}\, Q(g, g)(\lambda v) \tag{1.13}$$

(which is obtained by a homothetic change of variable). Equation (1.12) then implies that $V'(t) = \mathrm{cst} =: c_* > 0$. When the rescaled solution g does depend on time, its evolution equation is

$$T'(t)\, V(t)\, \partial_t g = Q(g, g) - c_*\, \nabla_v (v\, g). \tag{1.14}$$

We then choose T such this equation is as simple as possible: $T'(t)\, V(t) = 1$. Hence we deduce the natural choice of the scaling functions

$$K(t) = (1 + c_*\, t)^N, \quad T(t) = \frac{1}{c_*} \ln(1 + c_*\, t) \quad V(t) = (1 + c_*\, t) \tag{1.15}$$

for some constant $c_* > 0$. It is obvious that changing c_* in the Eq. (1.14) only amounts to the multiplication of g by a positive constant and the multiplication of T' by a positive constant. In the sequel we fix without restriction $c_* = 1$.

Summarizing, thanks to the Eq. (1.12) and to the rescaled variables defined by (1.11), (1.15), for any *self-similar profile* G solution to the stationary equation

$$Q(G, G) - \nabla_v \cdot (vG) = 0 \tag{1.16}$$

we may associate a *self-similar solution* F to the original Eq. (1.1) by setting

$$F(t, v) = (1 + t)^N\, G((1 + t)v).$$

Moreover, G is obviously a *stationary solution* to the rescaled evolution equation

$$\frac{\partial g}{\partial t} = Q(g, g) - \nabla_v \cdot (vg) \tag{1.17}$$

which is the equation associated to (1.1) making the change of variables (1.10), (1.15) (with $c_* = 1$). Roughly speaking the re-scaling (1.10), (1.15) adds an anti-drift to the original Eq. (1.1).

More generally, for any solution g to the Boltzmann equation in self-similar variables (1.17), we associate a solution f to the evolution problem (1.1), defining f by the relation

$$f(t, v) = (1 + t)^N \, g(\ln(1 + t), (1 + t)v). \tag{1.18}$$

Reciprocally, for any solution f to the Boltzmann equation (1.1), we associate a solution g to the evolution problem (1.17), defining g by the relation

$$g(t, v) = e^{-Nt} \, f(e^t - 1, e^{-t} v). \tag{1.19}$$

Given an initial datum $f_{\rm in} = g_{\rm in} \in L^1_2$, we know from Ref. 28 that there exists a unique solution of (1.1) in $C(\mathbb{R}_+, L^1_2) \cap L^1(\mathbb{R}_+, L^1_3)$. Therefore, thanks to the changes of variables (1.18), (1.19), we deduce that there exists a unique solution g to (1.17) in $C(\mathbb{R}_+, L^1_2) \cap L^1_{\rm loc}(\mathbb{R}_+, L^1_3)$. Moreover we have the following relations between the moments of f and g:

$$\forall\, t \geq 0, \qquad \begin{cases} \|g(t, \cdot) \, | \cdot |^k \|_{L^1} = e^{k\,t} \, \| f(e^t - 1, \cdot) \, | \cdot |^k \|_{L^1} \\[2mm] \| f(t, \cdot) \, | \cdot |^k \|_{L^1} = (1 + t)^{-k} \, \|g(\ln(1 + t), \cdot) \, | \cdot |^k \|_{L^1}. \end{cases} \tag{1.20}$$

1.3. Motivation

The use of Boltzmann inelastic hard spheres-like models to describe dilute, rapid flows of granular media started with the seminal physics paper,[(24)] and a huge physics litterature has developed in the last twenty years. The study of granular systems in such regime is motivated by their unexpected physical behavior (with the phenomena of *collapse*—or "*cooling effect*"—at the kinetic level and *clustering* at the hydrodynamical level), their use to derive hydrodynamical equations for granular fluids, and their applications. Granular gases are composed of *macroscopic grains*, and not microscopic molecules like in rarefied gas dynamics. The grains have only contact interactions, which motivates physical modelization by hard spheres with inelastic dissipative features built in the collision mechanism. The model of inelasticity with a constant normal restitution coefficient studied in the present paper is one of the simplest such model (for a more elaborated model, see for instance the so-called *visco-elastic hard spheres model* in Ref. 11, as well as Ref. 28).

From the physical and mathematical viewpoint, works on the inelastic Boltzmann models have been first restricted to the so-called *inelastic Maxwell molecules model*, which can be interpreted as an approximation where the collision rate is replaced by a mean value independent on the relative velocity. Existence, uniqueness of solutions and the rate of decay of the kinetic energy were obtained in Ref. 6 for the *inelastic Maxwell molecules model* with constant normal restitution coefficient. The *Maxwell molecules model* is important because of its analytic simplifications (with regards to the hard spheres model) allowing to use powerful Fourier transform tools. Polynomial tail behaviors of the self-similar profiles have been formally computed in Ref. 17. Convergence to self-similarity has been established in Refs. 7, 8. In the all, the *inelastic Maxwell model* is well understood now. Similar results have also been obtained for some simplified non-linear friction models in Refs. 12, 25.

For the *inelastic hard spheres* model (with constant normal restitution coefficient), one easily sees from the discussion in Sec. 1.2 that the kinetic energy of self-similar solutions to Eq. (1.1) (assuming their existence) behaves like

$$\mathcal{E}(t) \underset{t\to\infty}{\sim} \frac{C}{t^2}.$$

It is natural to expect a similar behavior for the rate of decay of the temperature for the generic solutions to Eq. (1.1). This conjecture was made twenty years ago in the pioneering paper,[24] and this rate of decay for the temperature is therefore known as *Haff's law*. Such a law is a typical physical feature of inelastic hard spheres which does not hold for inelastic Maxwell molecules model (for which the temperature follows an exponential law). Let us emphasize that in Ref. 6 a *pseudo-Maxwell molecules model* was considered (multiplying the collision operator by some well-chosen scalar function of time), restoring Haff's law and still preserving the nice simplications of Maxwell molecules. However some important aspects of inelastic hard spheres such as the tail behavior are not preserved by this model.

On the basis of the study of the particular case of Maxwell molecules, Ernst and Brito[16] conjectured that self-similar solutions, when they exist, should attract any solution, in the sense of convergence of the rescaled solution. They also conjectured and formally computed some "over-populated" tail behaviors for the self-similar profile (namely decreasing slower than the Maxwellian) depending on the collision rate (see Ref. 17 for instance).

Recently the works[9,19] laid the first steps for a mathematical analysis of the inelastic hard spheres model (with constant normal restitution coefficient). In Ref. 9, the authors proved the existence of steady states and gave estimates showing the presence of over-populated tails for *diffusively excited inelastic hard spheres*, that is when one provides an input of kinetic energy to the system preventing the collapse, which is modelized by some $\Delta_v f$ term added to Eq. (1.1). In Ref. 9 *a priori* integral estimates on the tail of the steady state (assuming its existence) were established for the spatially homogeneous inelastic Boltzmann equation with various additional terms, such as a diffusion, or an anti-drift as in (1.17).

In the present paper, we prove, for spatially homogeneous inelastic hard spheres with constant normal restitution coefficient, the existence of smooth self-similar solutions, and we improve the estimates on their tails of Ref. 9 into pontwise ones. We also give a complete regularity study of the generic solutions in the rescaled variables, as well as estimates on their tails. In particular, we give the first mathematical proof of Haff's law and we show the algebraic decay of singularities.

In a forthcoming work,[29] we shall prove the uniqueness and the local stability of these self-similar solutions for a small inelasticity.

1.4. Notation

Throughout the paper we shall use the notation $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. We denote, for any $q \in \mathbb{R}$, the Banach space

$$L^1_q = \left\{ f : \mathbb{R}^N \mapsto \mathbb{R} \text{ measurable}\,; \quad \|f\|_{L^1_q} := \int_{\mathbb{R}^N} |f(v)|\, \langle v \rangle^q \, dv < +\infty \right\}.$$

More generally we define the weighted Lebesgue space $L^p_q(\mathbb{R}^N)$ ($p \in [1, +\infty]$, $q \in \mathbb{R}$) by the norm

$$\|f\|_{L^p_q(\mathbb{R}^N)} = \left[\int_{\mathbb{R}^N} |f(v)|^p \, \langle v \rangle^{pq} \, dv \right]^{1/p}$$

when $p < +\infty$ and

$$\|f\|_{L^\infty_q(\mathbb{R}^N)} = \text{supess}_{v \in \mathbb{R}^N} |f(v)| \, \langle v \rangle^q$$

when $p = +\infty$ (where supess denotes the essential supremum).

The weighted Sobolev space $W^{k,p}_q(\mathbb{R}^N)$ ($p \in [1, +\infty]$, $q \in \mathbb{R}$ and $k \in \mathbb{N}$) is defined by the norm

$$\|f\|_{W^{k,p}_q(\mathbb{R}^N)} = \left[\sum_{|s| \le k} \|\partial^s f\|^p_{L^p_q} \right]^{1/p}$$

where ∂^s denotes the partial derivative associated with the multi-index $s \in \mathbb{N}^N$. In the particular case $p = 2$ we denote $H^k_q = W^{k,2}_q$. Moreover this definition can be extended to H^s_q for any $s \ge 0$ by using the Fourier transform.

We also denote by $L^1_{\text{loc}}(\Omega)$ the space of locally integrable functions on a given set $\Omega \subset \mathbb{R}^N$, that is the space of measurable functions on Ω which are integrable on every compact subset of Ω. Finally, for $h \in \mathbb{R}^N$, we define the translation operator τ_h by

$$\forall\, v \in \mathbb{R}^N, \quad \tau_h f(v) = f(v - h),$$

and we shall denote by "C" various constants which do not depend on the collision kernel B.

1.5. Main Results

In this subsection we consider a normal restitution coefficient $e \in (0, 1)$ (except in dimension $N = 3$ where the case $e = 0$ can be included, see the discussions in the proofs).

First we state a result of existence of self-similar solutions.

Theorem 1.1. *For any mass $\rho > 0$, there exists a self-similar profile G with mass ρ and momentum 0:*

$$0 \le G \in L^1_2, \quad Q(G, G) = \nabla_v \cdot (v\, G), \quad \int_{\mathbb{R}^N} G \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

which moreover can be built in such a way that G is radially symmetric, $G \in C^\infty$ and

$$\forall\, v \in \mathbb{R}^N, \quad a_1 e^{-a_2|v|} \le G(v) \le A_1 e^{-A_2|v|}$$

for some explicit constants $a_1, a_2, A_1, A_2 > 0$.

Second we establish that Haff's law holds.

Theorem 1.2. *For any $p \in (1, +\infty)$, $\rho > 0$, and some initial datum $f_{\rm in}$ such that*

$$0 \le f_{\rm in} \in L^1_2 \cap L^p, \quad \int_{\mathbb{R}^N} f_{\rm in} \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

the associated solution of the Boltzmann equation (1.1,1.2) in $C(\mathbb{R}_+; L^1_2) \cap L^1(\mathbb{R}_+; L^1_3)$ satisfies Haff's law in the sense:

$$\forall\, t \ge 0, \quad \frac{m}{(1+t)^2} \le \mathcal{E}(t) \le \frac{M}{(1+t)^2} \tag{1.21}$$

for some explicit constants $m, M > 0$ depending on the collision kernel and the mass, kinetic energy and L^p norm of $f_{\rm in}$.

Third we give a more precise and general result on the regularity and asymptotic behavior of the solutions g to the rescaled Eq. (1.17) (note that the first point of Theorem 1.3 implies in particular the preceding theorem).

Theorem 1.3. *For any $p \in (1, \infty)$, $\rho > 0$, and some initial datum $g_{\rm in}$ such that*

$$0 \le g_{\rm in} \in L^1_2 \cap L^p, \quad \int_{\mathbb{R}^N} g_{\rm in} \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

the unique solution g in $C(\mathbb{R}_+; L^1_2) \cap L^1_{loc}(\mathbb{R}_+; L^1_3)$ *of (1.17) with initial datum* g_{in} *satisfies:*

(i) *It remains bounded in* L^p *for all times, with uniform bound as t goes to infinity. Similarly for any* $q \geq 0$, *if* $g_{in} \in L^p_q$, *then the solution remains bounded in* L^p_q *for all times with uniform bound as t goes to infinity. As for the Sobolev norms, for any* $s, q \geq 0$, *there is* $w > 0$ *such that if* $g_{in} \in H^s_{q+w}$, *then the solution remains bounded in* H^s_q *for all times, with uniform bound as t goes to infinity.*

(ii) *For any arbitrarily large* $s, q \geq 0$, *there exists* $\lambda > 0$ *and some decomposition* $g = g^S + g^R$ *of the solution g such that* $g_S \geq 0$ *and*

$$\sup_{t \geq 0} \left\| g^S_t \right\|_{H^s_q} < +\infty, \qquad \left\| g^R_t \right\|_{L^1_2} = O\left(e^{-\lambda t}\right).$$

(iii) *Concerning the tail behavior, we have the following lower and upper bounds: there are some explicit constants* $a_1, a_2 > 0$ *such that*

$$\forall\, v \in \mathbb{R}^N, \qquad \liminf_{t \to \infty} g(t, v) \geq a_1\, e^{-a_2 |v|},$$

and, for any $\tau > 0$ *and* $s \in [0, 1/2)$, *there are some explicit constants* $A_1, A_2 > 0$ *such that (appearance of exponential moments)*

$$\forall\, t \geq \tau, \qquad \int_{\mathbb{R}^N} g(t, v)\, e^{-A_1 |v|^s}\, dv \leq A_2.$$

Moreover similar integral upper bounds with $s \in [1/2, 1]$ *are uniformly propagated in time if they are satisfied for the initial datum.*

All the constants in this theorem can be computed in terms of the mass, kinetic energy and the different norms assumed on g_{in}, *and the parameters.*

Remark 1.4. *Note that point (ii) of Theorem 1.3 implies, by coming back to the original variables, that for an initial datum* $0 \leq f_{in} \in L^1_2 \cap L^p$, $p \in (1, \infty)$, *the unique associated solution of (1.1) in* $C(\mathbb{R}_+; L^1_2) \cap L^1(\mathbb{R}_+; L^1_3)$ *satisfies a similar decomposition as above, but where the remaining part decreases with polynomial and not exponential rate. Hence we have shown that the amplitude of the singularities decreases algebraically in the original variables.*

More precisely it is likely that, in the original variables, singularities far from 0 *decrease in fact exponentially fast, whereas those close to* 0 *cannot decrease faster than polynomially (due to the fact that the damping effect of the loss part of the collision operator degenerates at this point).*

1.6. Method of Proof

The main tool in this paper is the regularity theory of the collision operator: we show that its gain part satisfies similar regularity properties as in the elastic case.(10,26,27,32,35) Following the study in the elastic case in Ref. 32, we deduce uniform propagation of Lebesgue norms for the solutions in the rescaled variables (1.17).

A first consequence is that the temperature in the rescaled variables is uniformly bounded from below by some positive number as soon as the initial datum satisfies some L^p bound. Translating this estimate in the original variables, it proves Haff's law.

A second consequence is the existence of self-similar profiles (or steady states) for the rescaled Eq. (1.17), which provides existence of self-similar solutions for the original Eq. (1.1). This existence result is proved by the use of a consequence of Tykhonov's fixed point Theorem (see Theorem 4.1), which is an infinite dimensional (rough) version of Poincaré-Bendixon Theorem on dynamical systems, see for instance [3, Théorème 7.4] or Refs. 18, 19. It states that a semi-group on a Banach space $\mathcal{Y}$ with suitable continuity properties, and which stabilizes a nonempty convex weakly compact subset, has a steady state inside this subset. We apply it to the evolution semi-group of (1.17) in the Banach space $\mathcal{Y} = L^1_2$. The existence and continuity properties of the semi-group were proved in Ref. 28 and the nonempty convex weakly compact subset of nonnegative functions with fixed mass and momentum and bounded moments and L^p norm, $p \in (1, +\infty)$ (for some bound big enough) is stable along the flow thanks to the above uniform L^p bounds in the rescaled variables.

Still following the study in the elastic case in Ref. 32, we also deduce from the regularity properties of the collision operator the uniform propagation of Sobolev norms as well as the exponential decay of (the amplitude of) the singularities for the solutions in the rescaled variables (1.17). That straightforwardly implies the smoothness of self-similar profiles as well as the algebraic decay of (the amplitude of) singularities for solutions to the Cauchy problem in the original variables (1.1)–(1.2).

Let us now turn to the study of the tail behavior. On the one hand, we prove lower pointwise estimates on the self-similar profiles by the mean of some elementary maximum principles arguments inspired from Ref. 19. On the other hand, we prove explicit lower pointwise estimates on generic solutions in self-similar variables using the spreading effect of the evolution semi-group associated to (1.17) (in the spirit of Refs. 13, 31, 33). Finally, upper pointwise estimates on the self-similar profiles are obtained using moments estimates established in Ref. 9 and elementary o.d.e. arguments.

1.7. Weak and Strong Forms of the Collision Operator

Under our assumptions on b, the function $\sigma \mapsto b(\hat{u} \cdot \sigma)$ is integrable on the sphere $\mathbb{S}^{N-1}$, and we can set without restriction

$$\int_{\mathbb{S}^{N-1}} b(\hat{u} \cdot \sigma)\, d\sigma = |\mathbb{S}^{N-2}| \int_0^{\pi} b(\cos\theta)\, \sin^{N-2}\theta\, d\theta = 1.$$

Thus we can write the classical splitting $Q = Q^+ - Q^-$ between gain part and loss part. The loss part Q^- is

$$Q^-(g, f)(v) := \left(\int_{\mathbb{R}^N} g(v_*)\, |v - v_*|\, dv_* \right) f(v) = (g * \Phi) f, \qquad (1.22)$$

where Φ denotes $\Phi(z) = |z|$. For any distribution g satifying the moment conditions $\int_{\mathbb{R}^N} g\, dv = 1$, $\int_{\mathbb{R}^N} g\, v\, dv = 0$, we have (see for instance [28, Lemma 2.2])

$$(g * \Phi) \geq |v|. \qquad (1.23)$$

The gain part Q^+ is defined by

$$Q^+(g, f)(v) := \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \frac{'f\, 'g_*}{e^2}\, |u|\, b(\hat{u} \cdot \sigma)\, d\sigma\, dv_*. \qquad (1.24)$$

In the sequel, we shall need two other representations. On the one hand from Ref. 14, there holds: for any $\psi \in L^\infty_1$, $f, g \in L^1_2$

$$\int_{\mathbb{R}^N} Q^+(g, f)(v)\, \psi(v)\, dv = \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} f\, g_*\, |u|\, b(\hat{u} \cdot \sigma)\, \psi(v')\, d\sigma\, dv_*\, dv, \qquad (1.25)$$

where v' denotes the post-collisional velocity defined by

$$v' = \frac{v + v_*}{2} + \frac{u'}{2}, \qquad u' = \frac{1-e}{2}\, u + \frac{1+e}{2}\, |u|\, \sigma. \qquad (1.26)$$

On the other hand, we shall establish a Carleman type representation for granular gases:

Proposition 1.5. *Let $E^e_{v,'v}$ be the hyperplan orthogonal to the vector $v - 'v$ and passing through the point $\Omega(v, 'v)$, defined by*

$$\Omega(v, 'v) := v + (1 - \beta^{-1})\, (v - 'v) = (2 - \beta^{-1}) v + (\beta^{-1} - 1)' v.$$

Then we have the following representation of the gain term

$$Q^+(g, f)(v) = \frac{2^{N-1}}{\beta^{N-1} e^2} \int_{'v \in \mathbb{R}^N} \int_{'v_* \in E^e_{v,'v}} |v - v_*|^{2-N} \times B\, |'v - v|^{-1}\, 'g_*\, 'f\, d'v\, dE('v_*). \qquad (1.27)$$

Recall that $\beta = (1+e)/(2e)$ and $B := B(u,\sigma) = |u|\, b(\hat{u}\cdot\sigma)$.

Remark 1.6. *Let us emphasize that in dimension $N = 3$ the expression (1.27) of the gain term Q^+ makes sense even when $e = 0$ (since $(e\beta)^2$ converges to $1/4$ and $\Omega(v, {}'v) = 2\,v - {}'v$ for $e = 0$), while the formula defining Q^+ in (1.3) seems to be singular when $e \to 0$. Hence this Carleman representation allows to define a strong formulation of Q^+ for $e = 0$, at least in the physical case of the dimension $N = 3$.*

Proof of Proposition 1.5. We start from the basic identity

$$\frac{1}{2}\int_{\mathbb{S}^{N-1}} F(|u|\sigma - u)\, d\sigma = \frac{1}{|u|^{N-2}} \int_{\mathbb{R}^N} \delta(2\,x\cdot u + |x|^2)\, F(x)\, dx, \qquad (1.28)$$

which can be verified easily by completing the square in the Dirac function, taking the spherical coordinate $x + u = r\,\sigma$ and performing the change of variable $r^2 = s$. We have the following relations from (1.4)

$$\begin{cases} {}'v = v + (\beta/2)\ (|u|\sigma - u) \\ {}'v_* = v_* - (\beta/2)\ (|u|\sigma - u) \end{cases} \qquad (1.29)$$

and thus starting from the strong form of Q^+ we get

$$Q^+(g,f) = e^{-2}\int_{\mathbb{R}^N\times\mathbb{S}^{N-1}} B\, f(v + (\beta/2)\ (|u|\sigma - u))\\ g(v_* - (\beta/2)\ (|u|\sigma - u))\, dv_*\, d\sigma.$$

Applying (1.28) yields

$$Q^+(g,f) = 2\,e^{-2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^{2-N}\, B\, \delta(2\,x\cdot u + |x|^2)\\ f(v + (\beta/2)x) g(v_* - (\beta/2)x)\, dv_*\, dx.$$

We do the change of variable $x \to {}'v = v + (\beta/2)x$ (with jacobian $(\beta/2)^N$). Then, keeping ${}'v$ fixed, we make the change of variable $v_* \to {}'v_*$ (with jacobian 1 since ${}'v_* = v + v_* - {}'v$). This gives

$$Q^+(g,f) = \frac{2^{N+1}}{\beta^N e^2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^{2-N}\, B\, \delta(2\,x\cdot u + |x|^2)\, f({}'v) g({}'v_*)\, d{}'v_*\, d{}'v.$$

Finally, keeping ${}'v$ fixed, we decompose orthogonally the variable ${}'v_*$ as $v + V_1\,n + V_2$ with $V_1 = ({}'v_* - v)\cdot n$, $n = ({}'v - v)/|{}'v - v|$ and V_2 orthogonal to $({}'v - v)$. Let us compute the Dirac function in the new coordinates. Since $x = (2/\beta)\,({}'v - v)$

and $u = (v - v_*)$,

$$2\,x \cdot u + |x|^2 = (4/\beta)\,('v - v) \cdot (v - v_*) + (4/\beta^2)\,|'v - v|^2$$
$$= (4/\beta)\,((\beta^{-1} - 1)\,|'v - v|^2 + ('v - v) \cdot ('v - v_*)).$$

From the momentum conservation $('v - v_*) = (v - {}'v_*)$ and the orthogonal decomposition above:

$$2\,x \cdot u + |x|^2 = (4/\beta)\,((\beta^{-1} - 1)\,|'v - v|^2 - V_1\,|'v - v|).$$

Hence we obtain the following representation:

$$Q^+(g, f) = \frac{2^{N+1}}{\beta^N e^2} \int_{\mathbb{R}\times\mathbb{R}^{N-1}\times\mathbb{R}^N} |u|^{2-N}\, B$$
$$\delta\left(\frac{4|'v - v|}{\beta}[(\beta^{-1} - 1)|'v - v| - V_1]\right) f('v) g(v + V_1\, n + V_2)\, dV_1\, dV_2\, d'v.$$

It remains to remove the Dirac mass: we use the obvious identity

$$\int_{\mathbb{R}} \delta\left(\frac{4|'v - v|}{\beta}[(\beta^{-1} - 1)|'v - v| - V_1]\right)$$
$$F(V_1)\, dV_1 = \frac{\beta}{4\,|'v - v|}\, F((\beta^{-1} - 1)|'v - v|)$$

to finally obtain representation (1.27). □

The parametrization by the Carleman representation means that for v and $'v$ fixed, the point $'v_*$ describes the hyperplan orthogonal to $('v - v)$ and passing through the point $\Omega(v, 'v)$ on the line determined by v and $'v$. Note that in the elastic case, $\Omega(v, 'v) = v$, whereas here $\Omega(v, 'v)$ is outside the segment $[v, 'v]$, which reflects the fact that for the pre-collisional velocities, the modulus of the relative velocity is bigger than $|v - v_*|$. In the limit case $e = 0$, $\Omega(v, 'v) = 2\,v - {}'v$.

The geometrical picture (in a plane section) is summerized in Fig. 1.

From this proposition we immediately deduce the following representation, which is closer to the classical Carleman representation for the elastic Boltzmann collision operator. From (1.29) we deduce

$$|'v - v| = \frac{\beta}{\sqrt{2}}\,|u|\,\sqrt{1 - (\hat{u} \cdot \sigma)}.$$

Hence we get

$$Q^+(f, g)(v) = C_e \int_{\mathbb{R}^N} \frac{'f}{|v - {}'v|^{N-2}} \left\{ \int_{E^e_{v,'v}} {}'g_*\, \tilde{b}(\hat{u} \cdot \sigma)\, d'v_* \right\} d'v,$$

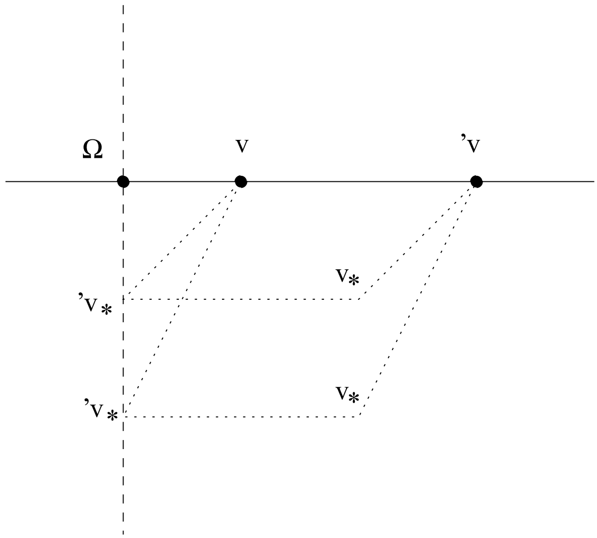

Fig. 1. Carleman representation for granular gases.

with

$$C_e = \frac{2^{\frac{3N-5}{2}}}{\beta^{2N-4} e^2} \qquad \text{and} \qquad \tilde{b}(x) = (1-x)^{-(N-3)}\, b(x).$$

2. REGULARITY PROPERTIES OF THE COLLISION OPERATOR

In this section the final goal is to estimate quantities such as

$$\int_{\mathbb{R}^N} Q(f,f)\, f^{p-1}\, dv$$

for $p > 1$, *i.e.*, the action of the collision operator on the evolution of the L^p norm (to the power p) of the solution along the flow. We shall use minoration estimates on Q^- deduced from (1.22)–(1.23), together with convolution and regularity estimates on Q^+. The latters seem to be new in the inelastic framework but they are an extension of similar estimates in the elastic case $e = 1$.

The estimates on Q^+ can be splitted into tree groups. First the convolution-like estimates, which originated (in the elastic case) in the works of Gustafsson[21,22] (see also Refs. 15, 28, 32) and were first extended to the inelastic case in Ref. 19. Second the regularization estimates in Sobolev spaces which originated in the elastic case in the works of Lions,[26] Bouchut and Desvillettes,[10] Lu [27] (see also Ref. 32 for some extensions). Third the non-concentration estimates in L^1, which originated in the elastic case in the work of Mischler and Wennberg,[30] and were extended by Abrahamsson.[1] Let us also mention that regularity properties of Q^+ are reminiscent of the work of Grad on the linearized collision operator.[23] The main tool to extend the second group of estimates to the inelastic case shall be the Carleman representation for granular gases of Proposition 1.5 (the third group of estimates can be extended with this tool as well, see Ref. 29). Before turning to the regularity study of Q^+, we recall convolution-like estimates.

2.1. Convolution-Like Estimates

In the elastic case $e = 1$, convolution-like estimates for the gain part of the collision operator were first proved in Refs. 21, 22. This proof was simplified by a duality argument in Ref. 32, where also a more precise statement was given. These estimates were extended to the inelastic case, for a constant normal restitution coefficient $e \in [0, 1]$, in Ref. 19 (in a form slightly less precise than in Ref. 32). Also a result weaker in one aspect (less precise for the treatment of the algebraic weight) but more general in another (valid in any Orlicz spaces, and valid for more general collision kernels) was proved in Ref. 28. Here we only state the precise result we shall need, whose proof is straightforward from the arguments in [19, Proof of Lemma 4.1] and [32, Proof of Theorem 2.1].

We make the following assumption on the cross-section: no *frontal collision* should occur, *i.e.*, $b(\cos\theta)$ should vanish for θ close to π:

$$\exists\, \theta_b > 0\,; \quad \text{support} \quad b\,(\cos\theta) \subset \{\theta \;/\; 0 \le \theta \le \pi - \theta_b\}\,. \tag{2.1}$$

To exchange the roles of f and g, we introduce the symmetric assumption that no *grazing collision* should occur, *i.e.*,

$$\exists\, \theta_b > 0\,; \quad \text{support} \quad b\,(\cos\theta) \subset \{\theta \;/\; \theta_b \le \theta \le \pi\}\,. \tag{2.2}$$

Then we have (from the proofs of [19, Lemma 4.1 and Proposition 4.2]):

Theorem 2.1. *Let $k, \eta \in \mathbb{R}$, $p \in [1, +\infty]$, and let $B = \Phi\, b$ be a collision kernel with b satisfying the assumption (2.1). Then for any $e \in [0, 1]$ the associated gain term satisfies the estimates*

$$\left\| Q^+(g, f)\right\|_{L^p_\eta} \le C_{k,\eta,p}(B)\, \|g\|_{L^1_{|k+\eta|+|\eta|}}\, \|f\|_{L^p_{k+\eta}}\,,$$

with

$$C_{k,\eta,p}(B) = C\, (\sin(\theta_b/2))^{\min(\eta,0)-2/p'}\; \|b\|_{L^1(\mathbb{S}^{N-1})}\, \|\Phi\|_{L^\infty_{-k}}\,.$$

If on the other hand assumption (2.1) is replaced by assumption (2.2), then the same estimates hold with $Q^+(g, f)$ replaced by $Q^+(f, g)$.

2.2. Lions Theorem for Q^+

In this subsection we assume that the collision kernel $B = \Phi\, b$ satisfies

$$\Phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}), \qquad b \in C_0^\infty(-1, 1). \tag{2.3}$$

Then we have the

Theorem 2.2. *Let B be a collision kernel satisfying (2.3). Then for any $e \in (0, 1]$, the associated gain term Q^+ satisfies for all $s \in \mathbb{R}_+$ and $\eta \in \mathbb{R}_+$*

$$\|Q^+(g, f)\|_{H^{s+(N-1)/2}_\eta} \le C(s, B)\, \|g\|_{H^s_\eta}\, \|f\|_{L^1_{2\eta}}$$

for some explicit constants $C(s, B) > 0$ depending only on s and the collision kernel.

Remark 2.3. *In dimension $N = 3$ this theorem extends to the case $e = 0$, with uniform bound $C(s, B)$ for $e \in [0, 1]$. The only obstacle to the treatment of the case $e = 0$ indeed is the constant $2^{N-1}\, \beta^{-(N-1)}\, e^{-2}$ in front of the Carleman representation, which may blow up as $e \to 0$.*

Proof of Theorem 2.2. We follow closely the proof of Ref. 32, inspired from the works of Lions[26] and Wennberg.[35] Indeed the Carleman representation proved above in Proposition 1.5 allows essentially to reduce to the study of the elastic case.

We assume first that $\eta = 0$. We denote

$$\mathcal{B}(|'v - 'v_*|, |'v - v|) = \frac{B(|v - v_*|, \cos\theta)}{|v - v_*|^{N-2} |'v - v|}$$

which belongs to $C_0^\infty((\mathbb{R}_+ \setminus \{0\})^2)$ under assumption (2.3). We define the following (Radon transform type) functional: for g smooth enough, Tg is defined by

$$Tg(y) = \int_{\mu y + y^\perp} \mathcal{B}(z, y)\, g(z)\, dz$$

with $\mu = (2 - \beta^{-1})$. Let us relate this functionnal with the Carleman representation (1.27): we have

$$\begin{aligned}
&\int_{'v_* \in E^e_{v,'v}} |v - v_*|^{2-N}\, B\, |'v - v|^{-1}\ 'g_*\, d'v_* \\
&\quad = \int_{'v_* \,\in\, \Omega(v,\,'v) + (v - 'v)^\perp} \mathcal{B}(|'v_* - 'v|, |v - 'v|)\ \ 'g_*\, d'v_* \\
&\quad = \int_{z \,\in\, \Omega(v,\,'v) - 'v + (v - 'v)^\perp} \mathcal{B}(|z|, |v - 'v|)\ \ \tau_{-'v} g(z)\, dz,
\end{aligned}$$

and since

$$\Omega(v, {}'v) - {}'v = (2 - \beta^{-1})\,(v - {}'v),$$

we deduce

$$\int_{'v_* \in E^e_{v,'v}} |v - v_*|^{2-N}\, B\, |{}'v - v|^{-1}\; {}'g_*\, d'v_*$$
$$= \int_{\mu\,(v-{}'v)+(v-{}'v)^{\perp}} \mathcal{B}(|z|, |v - {}'v|)\;\; \tau_{-{}'v} g(z)\, dz$$
$$= (\tau_{'v} \circ T \circ \tau_{-{}'v})\,(g)(v).$$

Hence the representation (1.27) writes

$$Q^{+}(g, f)(v) = \frac{2^{N-1}}{\beta^{N-1}\, e^2} \int_{\mathbb{R}^N} f({}'v)\; (\tau_{'v} \circ T \circ \tau_{-{}'v})\,(g)(v)\, d'v.$$

Thus if one has a bound on T of the form

$$\|Tg\|_{H^{s+(N-1)/2}} \le C_T\, \|g\|_{H^s}, \quad C_T > 0, \tag{2.4}$$

then by using Fubini's and Jensen's theorems one gets

$$\|Q^{+}(g, f)\|^2_{H^{s+(N-1)/2}} \le C\, \|f\|_{L^1} \int_{\mathbb{R}^N} f({}'v)\; \|(\tau_{'v} \circ T \circ \tau_{-{}'v})\,(g)\|^2_{H^{s+(N-1)/2}}\; d'v$$
$$\le C\, \|f\|_{L^1} \int_{\mathbb{R}^N} f({}'v)\; \|(T \circ \tau_{-{}'v})\,(g)\|^2_{H^{s+(N-1)/2}}\; d'v$$
$$\le C\, C_T\, \|f\|_{L^1} \int_{\mathbb{R}^N} f({}'v)\; \|\tau_{-{}'v} g\|^2_{H^s}\; d'v$$
$$\le C\, C_T\; \|g\|^2_{H^s}\; \|f\|_{L^1} \int_{\mathbb{R}^N} f({}'v)\, d'v \le C\, C_T\; \|g\|^2_{H^s}\; \|f\|^2_{L^1},$$

which concludes the proof. Thus it remains to prove (2.4). But, up to an homothetic factor, T is exactly the operator which was studied in detail in Refs. 32, 35. More precisely,

$$Tg(y) = \tilde{T} g(\mu y)$$

where $\tilde{T}$ is the Radon transform

$$\tilde{T} g(y) = \int_{y + y^{\perp}} \tilde{\mathcal{B}}(z, y)\, g(z)\, dz,$$

which was introduced in the elastic case in Ref. 35, associated with a kernel $\tilde{\mathcal{B}}$ related to our collision kernel by

$$\tilde{\mathcal{B}}(z, y) = \mathcal{B}(z, \mu^{-1}\, y).$$

It was proved in [32, Proof of Theorem 3.1] that

$$\|\tilde{T} g\|_{H^{s+(N-1)/2}} \leq C\, \|g\|_{H^s}$$

for an explicit bound C depending on some weighted Sobolev norms on $\tilde{\mathcal{B}}$. Coming back to T, we obtain (2.4). This ends the proof when $\eta = 0$. The extension to $\eta > 0$ is straightforward (and exactly similar to [32, Proof of Theorem 3.1]). □

As a Corollary we deduce from Theorem 2.2 the following estimate in Lebesgue spaces by Sobolev embeddings (the proof is exactly similar to [32, Proof of Corollary 3.2]).

Corollary 2.4. *Let B be a collision kernel satisfying (2.3). Then, for all $p \in (1, +\infty)$, $\eta \in \mathbb{R}$, we have*

$$\left\| Q^+(g, f)\right\|_{L^p_\eta} \leq C(p, \eta, B)\, \|g\|_{L^q_\eta}\, \|f\|_{L^1_{2|\eta|}}$$

where the constant $C(p, \eta, B) > 0$ only depends on the collision kernel, p and η, and $q < p$ is given by

$$q = \begin{cases} \dfrac{(2N-1)p}{N+(N-1)p} & \textit{if } p \in (1, 2N] \\ \dfrac{p}{N} \quad \textit{if } p \in [2N, +\infty). \end{cases} \tag{2.5}$$

2.3. Bouchut-Desvillettes-Lu Theorem on Q^+

Now we turn to a slightly different regularity estimate on Q^+, which is a straightforward extension of the works(10,27) in the elastic case $e = 1$. This class of estimate is weaker than Lions's Theorem 2.2 since the Sobolev norm of Q^+ is controlled by the square of the Sobolev norm of the solution with smaller order, which does not allow to take advantage of the L^1 theory. Nevertheless, it is more convenient in other aspects since it deals directly with the physical collision kernel.

Theorem 2.5. *Under the assumptions made on B in Section 1.1, for any $e \in [0, 1]$ the associated gain term Q^+ satisfies, for all $s \in \mathbb{R}_+$ and $\eta \in \mathbb{R}_+$,*

$$\|Q^+(g, f)\|_{H^{s+(N-1)/2}_\eta} \leq C(s, B)\, [\|g\|_{H^s_{\eta+2}} \|f\|_{H^s_{\eta+2}} + \|g\|_{L^1_{\eta+2}} \|f\|_{L^1_{\eta+2}}]$$

for some explicit constant $C(s, B) > 0$ depending only on s and B.

Proof of Theorem 2.5. We follow closely the method in Ref. 10. We write it for $\eta = 0$ but the general case is strictly similar.

Let us denote $F(v, v_*) = f(v)\, g(v_*)\, |v - v_*|$. The same arguments as in Ref. 10 easily lead to

$$\mathcal{F}Q^+(\xi) = \int_{\mathbb{S}^{N-1}} \widehat{F}(\xi^+, \xi^-)\, b(\hat{\xi} \cdot \sigma)\, d\sigma$$

where $\mathcal{F}Q^+$ denotes the Fourier transform of Q^+ according to v, $\widehat{F}$ denotes the Fourier transform of F according to v, v_*, and

$$\xi^+ = \frac{3-e}{4}\, \xi + \frac{1+e}{4}\, |\xi| \sigma, \qquad \xi^- = \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, |\xi| \sigma.$$

Thus

$$|\mathcal{F}Q^+(\xi)|^2 \le \|b\|^2_{L^2(\mathbb{S}^{N-1})} \left(\int_{\mathbb{S}^{N-1}} |\widehat{F}(\xi^+, \xi^-)|^2\, d\sigma \right).$$

Let us consider frequencies ξ such that $|\xi| \ge 1$. As

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}} |\widehat{F}(\xi^+, \xi^-)|^2\, d\sigma \\
&\quad = \int_{\mathbb{S}^{N-1}} \int_{|\xi|}^{+\infty} -\frac{\partial}{\partial r} \left| \widehat{F}\left(\frac{3-e}{4}\, \xi + \frac{1+e}{4}\, r\sigma, \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, r\sigma \right) \right|^2 d\sigma\, dr \\
&\quad \le C \int_{\mathbb{S}^{N-1}} \int_{|\xi|}^{+\infty} \left| \widehat{F}\left(\frac{3-e}{4}\, \xi + \frac{1+e}{4}\, r\sigma, \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, r\sigma \right) \right| \\
&\qquad \times \left| (\nabla_2 - \nabla_1) \widehat{F}\left(\frac{3-e}{4}\, \xi + \frac{1+e}{4}\, r\sigma, \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, r\sigma \right) \right| d\sigma\, dr \\
&\quad \le C \int_{|\zeta| \ge |\xi|} \left| \widehat{F}\left(\frac{3-e}{4}\, \xi + \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, \zeta \right) \right| \\
&\qquad \times \left| (\nabla_2 - \nabla_1) \widehat{F}\left(\frac{3-e}{4}\, \xi + \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, \zeta \right) \right| \frac{d\zeta}{|\zeta|^{N-1}},
\end{aligned}$$

where we have made the spherical change of variable $\zeta = r\sigma$, we deduce

$$\begin{aligned}
&\int_{|\xi| \ge 1} |\mathcal{F}Q^+(\xi)|^2\, |\xi|^{2s+(N-1)}\, d\xi \\
&\quad \le C\, \|b\|^2_{L^2(\mathbb{S}^{N-1})} \int_{1 \le |\xi| \le |\zeta|} \left| \widehat{F}\left(\frac{3-e}{4}\, \xi + \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, \zeta \right) \right| \\
&\quad \times \left| (\nabla_2 - \nabla_1) \widehat{F}\left(\frac{3-e}{4}\, \xi + \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\, \xi - \frac{1+e}{4}\, \zeta \right) \right| \frac{|\xi|^{2s+(N-1)}}{|\zeta|^{N-1}}\, d\xi\, d\zeta.
\end{aligned}$$

Finally we make the change of variable

$$X = \frac{3-e}{4}\,\xi + \frac{1+e}{4}\,\zeta, \qquad Y = \frac{1+e}{4}\,\xi - \frac{1+e}{4}\,\zeta,$$

(whose Jacobian is uniformly bounded from above and below for $e \in [0, 1]$) to obtain

$$\begin{aligned}\int_{|\xi|\geq 1} |\mathcal{F}Q^+(\xi)|^2 |\xi|^{2s+(N-1)}\, d\xi &\leq C\, \|b\|^2_{L^2(\mathbb{S}^{N-1})} \int_{\mathbb{R}^N\times\mathbb{R}^N} \left|\widehat{F}(X,Y)\right| \\ &\quad \times \left|(\nabla_2 - \nabla_1)\widehat{F}(X,Y)\right| \langle X\rangle^{2s} \langle Y\rangle^{2s}\, dX\, dY \\ &\leq C\, \|b\|^2_{L^2(\mathbb{S}^{N-1})} \|F\|_{H^s} \|(v-v_*)F\|_{H^s} \\ &\leq C\, \|b\|^2_{L^2(\mathbb{S}^{N-1})} \|g\|^2_{H^s_2} \|f\|^2_{H^s_2}.\end{aligned}$$

Then small frequencies are controlled thanks to the L^1 norms of f and g, which concludes the proof. □

2.4. Estimates on the Global Collision Operator in Lebesgue Spaces

We consider a collision kernel $B = \Phi\, b$ with $\Phi(u) = |u|$ and b integrable. We shall make a splitting of Q^+ as in [32, Sec. 3.1]. We denote by $\mathbf{1}_E$ the usual indicator function of the set E.

Let $\Theta : \mathbb{R} \to \mathbb{R}_+$ be an even C^∞ function such that support $\Theta \subset (-1, 1)$, and $\int_{\mathbb{R}} \Theta\, dx = 1$. Let $\widetilde{\Theta} : \mathbb{R}^N \to \mathbb{R}_+$ be a radial C^∞ function such that support $\widetilde{\Theta} \subset B(0, 1)$ and $\int_{\mathbb{R}^N} \widetilde{\Theta}\, dx = 1$. Introduce the regularizing sequences

$$\begin{cases} \Theta_m(x) = m\, \Theta(mx), & x \in \mathbb{R}, \\ \widetilde{\Theta}_n(x) = n^N \widetilde{\Theta}(nx), & x \in \mathbb{R}^N. \end{cases}$$

We use these mollifiers to split the collision kernel into a smooth and a non-smooth part. As a convention, we shall use subscripts S for "smooth" and R for "remainder". First, we set

$$\Phi_{S,n} = \widetilde{\Theta}_n * \left(\Phi\, \mathbf{1}_{\mathcal{A}_n}\right), \qquad \Phi_{R,n} = \Phi - \Phi_{S,n},$$

where $\mathcal{A}_n$ stands for the annulus $\mathcal{A}_n = \left\{x \in \mathbb{R}^N \,;\, \frac{2}{n} \leq |x| \leq n\right\}$. Similarly, we set

$$b_{S,m}(z) = \Theta_m * \left(b\, \mathbf{1}_{\mathcal{I}_m}\right)(z), \qquad b_{R,m} = b - b_{S,m},$$

where $\mathcal{I}_m$ stands for the interval $\mathcal{I}_m = \left\{x \in \mathbb{R} \,;\, -1 + \frac{2}{m} \leq |x| \leq 1 - \frac{2}{m}\right\}$ (b is understood as a function defined on $\mathbb{R}$ with compact support in $[-1, 1]$). Finally, we set

$$Q^+ = Q^+_S + Q^+_R,$$

where

$$Q_S^+(g, f) = e^{-2} \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi_{S,n}(|v - v_*|)\, b_{S,m}(\cos\theta)\, 'g_*\, 'f \, d\sigma\, dv_*$$

and

$$Q_R^+ = Q_{RS}^+ + Q_{SR}^+ + Q_{RR}^+$$

with the obvious notation

$$\begin{cases} Q_{RS}^+(g, f) = e^{-2} \displaystyle\int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi_{R,n}\, b_{S,m}\, 'g_*\, 'f \, dv_*\, d\sigma \\ Q_{SR}^+(g, f) = e^{-2} \displaystyle\int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi_{S,n}\, b_{R,m}\, 'g_*\, 'f \, dv_*\, d\sigma \\ Q_{RR}^+(g, f) = e^{-2} \displaystyle\int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi_{R,n}\, b_{R,m}\, 'g_*\, 'f \, dv_*\, d\sigma. \end{cases}$$

Now we follow the proof as in [32, Sec. 4.1] since we have the same functional inequalities in Sobolev and Lebesgue spaces, using also some ideas from Refs. 15, 28 to simplify it.

Proposition 2.6. *Let us consider $e \in (0, 1]$ and the associated gain term Q^+. For any $\varepsilon > 0$, there exists $\theta \in (0, 1)$, only depending on N and p, and a constant $C_\varepsilon > 0$, only depending on N, p, B and ε (and blowing up as $\varepsilon \to 0$), such that*

$$\int_{\mathbb{R}^N} Q^+(f, f)\, f^{p-1}\, dv \le C_\varepsilon\, \|f\|_{L^1}^{1+p\theta}\, \|f\|_{L^p}^{p(1-\theta)} + \varepsilon\, \|f\|_{L^1_2}\, \|f\|_{L^p_{1/p}}^p .$$

Remark 2.7. *For general dimension $N \ge 2$, the estimates in this section are valid only for $e \in (0, 1]$ (the constants may blow up as $e \to 0$). However when $N = 3$ they are uniform on $e \in (0, 1]$ and then extend to the limit case $e = 0$ (see also Remark 1.6).*

Proof of Proposition 2.6. Let us fix $\varepsilon > 0$. We split Q^+ as $Q_S^+ + Q_{RS}^+ + Q_{SR}^+ + Q_{RR}^+$ and we estimate each term separately. Remember that the truncation parameters n (for the kinetic part) and m (for the angular part) are implicit in the decomposition of Q^+.

By Corollary 2.4, there exists a constant $C(m, n) > 0$, blowing up as m or n goes to infinity, such that

$$\left\| Q_S^+(f, f)\right\|_{L^p} \le C(m, n)\, \|f\|_{L^q}\, \|f\|_{L^1},$$

for some $q < p$ defined in (2.5). Hence by Hölder's inequality,

$$\int_{\mathbb{R}^N} f^{p-1} Q_S^+(f,f)\,dv \le \left[\int_{\mathbb{R}^N} f^p\,dv\right]^{\frac{p-1}{p}} \left[\int_{\mathbb{R}^N} (Q_S^+)^p\,dv\right]^{\frac{1}{p}}$$

$$\le \|f\|_{L^p}^{p-1} \left\| Q_S^+(f,f)\right\|_{L^p} \le C(m,n)\,\|f\|_{L^q}\,\|f\|_{L^1}\,\|f\|_{L^p}^{p-1}.$$

Next we fix a weight $\eta \ge -1$ and we estimate the L^p_η norm of $Q^+_{SR}(f,f)$ and $Q^+_{RR}(f,f)$. We use that $\|b_{R,m}\|_{L^1(\mathbb{S}^{N-1})}$ goes to 0 as m goes to infinity (since b is integrable on the sphere), and we obtain, using Theorem 2.1 with $k=1$ and splitting the angular integration between a part $\hat{u}\cdot\sigma \le 0$ with no grazing collision and a part $\hat{u}\cdot\sigma \ge 0$ with no frontal collision,

$$\left\| Q^+_{SR}(f,f),\ Q^+_{RR}(f,f)\right\|_{L^p_\eta} \le \epsilon(m)\,\|f\|_{L^1_{|1+\eta|+|\eta|}}\,\|f\|_{L^p_{1+\eta}},$$

for some $\epsilon(m)$ going to 0 as m goes to infinity. Since $1+\eta \ge 0$, we can write $|1+\eta| + |\eta| = 1 + 2\eta_+$, where $\eta_+ = \max\{\eta, 0\}$. Hence by Hölder's inequality,

$$\int_{\mathbb{R}^N} f^{p-1}(Q^+_{SR}(f,f) + Q^+_{RR}(f,f))\,dv = \int_{\mathbb{R}^N} (f\langle v\rangle^{1/p})^{p-1}\,\frac{Q^+_{SR} + Q^+_{RR}}{\langle v\rangle^{\frac{1}{p'}}}\,dv$$

$$\le \left[\int_{\mathbb{R}^N} (f\langle v\rangle^{1/p})^p\,dv\right]^{\frac{p-1}{p}} \left[\int_{\mathbb{R}^N} ((Q^+_{SR} + Q^+_{RR})\langle v\rangle^{-1/p'})^p\,dv\right]^{\frac{1}{p}}$$

$$\le \|f\|^{p-1}_{L^p_{1/p}} (\|Q^+_{SR}(f,f)\|_{L^p_{-1/p'}} + \|Q^+_{RR}(f,f)\|_{L^p_{-1/p'}}) \le \epsilon(m)\,\|f\|_{L^1_1}\,\|f\|^p_{L^p_{1/p}}.$$

It remains to estimate the term corresponding to Q^+_{RS}. We have the trivial estimate

$$\Phi_{R,n} \le C\,n^{-1}\,(|v|^2 + |v_*|^2)$$

from which we deduce that

$$\begin{aligned} I &:= \int_{\mathbb{R}^N} f^{p-1} Q^+_{RS}(f,f)\,dv \\ &\le C\,n^{-1} \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^N} b_{S,m}\, f\, f_*\,(f')^{p-1}\,(|v|^2 + |v_*|^2)\,dv\,dv_*\,d\sigma := I_1 + I_2. \end{aligned}$$

Now we treat separately the two terms of the right-hand side:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^N} b_{S,m}\,(f\,|v|^2)\,f_*\,(f')^{p-1}\,dv\,dv_*\,d\sigma \\ &\le C \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^N} b_{S,m}\,\left[f_*^p + (f')^p\right]\,(f\,|v|^2)\,dv\,dv_*\,d\sigma := I_{1,1} + I_{1,2} \end{aligned}$$

using Young's inequality $xy^{p-1} \leq (1/p)\,x^p + ((p-1)/p)\,y^p$ on the product $f_*(f')^{p-1}$. The control of $I_{1,1}$ is immediate by integrating separately the angular variable:

$$I_{1,1} \leq C\,\|f\|_{L^1_2}\,\|f\|_{L^p}.$$

For $I_{1,2}$, as in the proof [19, Proposition 4.3] and using the notations of [28, Lemma 4.4], we make the change of variable $v_* \mapsto v' = \phi^*(v_*) = \phi^*_{e,v,\sigma}(v_*)$ keeping v, σ fixed, which is a C^∞-diffeomorphism from $\mathcal{O} = \{v_* \in \mathbb{R}^N, \hat{u}\cdot\sigma \neq 1\}$ onto its image. Thanks to [28, Lemma 4.4] and because $b_{S,m}$ has compact support in $(-1, 1)$, its Jacobian $J^* = \det(D\phi^*)$ satisfies

$$C(m)^{-1} \leq J^*(v_*) \leq C(m) \qquad \forall\, v_* \in \mathbb{R}^N,\ \hat{u}\cdot\sigma \in \text{support}\ b_{S,m}, \tag{2.6}$$

for some constant $C(m) \in (0, \infty)$ which blows up when m goes to ∞. Hence we straightforwardly deduce

$$I_{1,2} = C \int_{\mathbb{R}^N\times\mathbb{S}^N} \int_{\phi^*(\mathcal{O})} b_{S,m}\, \frac{(f')^p}{J^*\circ\phi^{*-1}(v')}$$
$$f\,|v|^2\, dv'\, dv\, d\sigma \leq C(m)\,\|f\|_{L^1_2}\,\|f\|_{L^p}.$$

The term I_2 is treated in a similar way: it is splitted as above using Young's inequality on the product $f(f')^{p-1}$. The term $I_{2,1}$ involving f^p is directly estimated as for the term $I_{1,1}$. For the term $I_{2,2}$ involving $(f')^p$ we proceed as for the term $I_{1,2}$. We make now the change of variable $v \mapsto v' = \phi_{e,v,\sigma}(v)$ keeping v_*, σ fixed, where we use again the notations and results of [28, Lemma 4.4]. Since its Jacobian J also satisfies the bound (2.6), we get the estimate (2.7) for the term $I_{2,2}$. We finally deduce

$$I \leq \frac{C(m)}{n}\,\|f\|_{L^1_2}\,\|f\|_{L^p}.$$

Gathering the previous estimates we deduce

$$\int_{\mathbb{R}^N} f^{p-1} Q^+(f,f)\,dv \leq C(m,n)\,\|f\|_{L^q}\,\|f\|_{L^1}\,\|f\|^{p-1}_{L^p}$$
$$+\frac{C(m)}{n}\,\|f\|_{L^1_2}\,\|f\|_{L^p} + \epsilon(m)\,\|f\|_{L^1_2}\,\|f\|^p_{L^p_{1/p}}$$

where q is defined by (2.5), $\epsilon(m)$ goes to 0 as m goes to infinity, and $C(m,n), C(m) > 0$. Hence for any given $\varepsilon > 0$, by first fixing m big enough, then n big enough, we get

$$\int_{\mathbb{R}^N} f^{p-1} Q^+(f,f)\,dv \leq C_\varepsilon\,\|f\|_{L^q}\,\|f\|_{L^1}\,\|f\|^{p-1}_{L^p} + \varepsilon\,\|f\|_{L^1_2}\,\|f\|^p_{L^p_{1/p}}$$

for some explicit constant $C_\varepsilon > 0$. Combining this with elementary interpolation, we deduce that there exists $\theta \in (0, 1)$, only depending on N and p, and a constant $C_\varepsilon > 0$, only depending on N, p, B and ε, such that

$$\int_{\mathbb{R}^N} f^{p-1} Q^+(f, f)\, dv \le C_\varepsilon \|f\|_{L^1}^{1+p\theta} \|f\|_{L^p}^{1-p\theta} \|f\|_{L^p}^{p-1} + \varepsilon \|f\|_{L^1_2} \|f\|^p_{L^p_{1/p}}$$
$$\le C_\varepsilon \|f\|_{L^1}^{1+p\theta} \|f\|_{L^p}^{p(1-\theta)} + \varepsilon \|f\|_{L^1_2} \|f\|^p_{L^p_{1/p}}.$$

This concludes the proof. □

3. REGULARITY STUDY IN THE RESCALED VARIABLES

In this section we show the uniform propagation of Lebesgue and Sobolev norms and the exponential decay of singularities for the solutions of (1.17).

3.1. Uniform Propagation of Moments - Povzner Lemma

Let us prove that the kinetic energy of g remains uniformly bounded from above as t goes to infinity. Using (1.17) and (1.9), we get

$$\frac{d}{dt} \int_{\mathbb{R}^N} g\, |v|^2\, dv \le -\tau \int_{\mathbb{R}^N \times \mathbb{R}^N} g\, g_*\, |u|^3\, dv_* dv + 2 \int_{\mathbb{R}^N} g\, |v|^2\, dv.$$

On the one hand, from Jensen's inequality (see for instance [28, Lemma 2.2]), there holds

$$\int_{\mathbb{R}^N} g_*\, |u|^3\, dv_* \ge \rho\, |v|^3.$$

On the other hand, Hölder's inequality yields

$$\int_{\mathbb{R}^N} g\, |v|^2\, dv \le \left(\int_{\mathbb{R}^N} g\, dv \right)^{1/3} \left(\int_{\mathbb{R}^N} g |v|^3\, dv \right)^{2/3},$$

which implies that

$$\int_{\mathbb{R}^N} g\, |v|^3\, dv \ge \rho^{1/2} \left(\int_{\mathbb{R}^N} g |v|^2\, dv \right)^{3/2}.$$

Thus

$$\frac{d}{dt} \int_{\mathbb{R}^N} g\, |v|^2\, dv \le -\tau\, \rho^{3/2} \left(\int_{\mathbb{R}^N} g\, |v|^2\, dv \right)^{3/2} + 2 \left(\int_{\mathbb{R}^N} g\, |v|^2\, dv \right)$$
$$\le \tau\, \rho^{3/2} \left(\int_{\mathbb{R}^N} g\, |v|^2\, dv \right) \left[\frac{2}{\tau \rho^{3/2}} - \left(\int_{\mathbb{R}^N} g\, |v|^2\, dv \right)^{1/2} \right],$$

and by maximum principle we deduce

$$\sup_{t\geq 0}\int_{\mathbb{R}^N} g\,|v|^2\,dv \leq C_E \max\left\{\left(\frac{4}{\rho^3\,\tau^2}\right), \int_{\mathbb{R}^N} g_{\rm in}\,|v|^2\,dv\right\}. \tag{3.1}$$

The same argument, together with sharp versions of Povzner inequalities from Refs. 5, 9, yields uniform bounds and appearance on every moments of the solution, as well as appearance of some exponential moments (this last point was first noticed in Ref. 28), in a similar way as in [28, Proof of Proposition 3.2]. Indeed we prove the.

Proposition 3.1. *Let g be a solution in $C(\mathbb{R}_+; L^1_2) \cap L^1_{\rm loc}(\mathbb{R}_+; L^1_3)$ to the rescaled Boltzman equation (1.17) with $e \in [0, 1)$, with initial datum $g_{\rm in}$. Then it satisfies the following additional moment properties:*

(i) For any $s \geq 2$, there is an explicit constant $C_s > 0$, depending only on B, e, and $g_{\rm in}$, such that

$$\sup_{t\in[0,\infty)} \|g(t,\cdot)\|_{L^1_s} \leq \max\{\|g_{\rm in}\|_{L^1_s}, C_s\}.$$

(ii) If $g_{\rm in}\,e^{r\,|v|^\eta} \in L^1(\mathbb{R}^N)$ for $r > 0$ and $\eta \in (0, 1]$, there exists $C_1, r' > 0$, depending only on B, e, and $g_{\rm in}$, such that

$$\sup_{t\in[0,\infty)} \int_{\mathbb{R}^N} g(t, v)\, e^{r'\,|v|^\eta}\,dv \leq C_1.$$

(iii) For any $\eta \in (0, 1/2)$ and $\tau > 0$, there exists $a_\eta, C_\eta \in (0, \infty)$, depending only on B, e, τ and $g_{\rm in}$, such that

$$\sup_{t\in[\tau,\infty)} \int_{\mathbb{R}^N} g(t, v)\, e^{a_\eta\,|v|^\eta}\,dv \leq C_\eta.$$

Let us emphasize that the constant C_s, a_η, C_η may depend on $g_{\rm in}$ only through its mass ρ and its kinetic energy $\mathcal{E}_{\rm in}$.

Proof of Proposition 3.1. The proof is just a copy with minor modifications of classical proofs. For the proof of (i) we refer for intance to Refs. 19, 30, 34 and the references therein. The proofs of (ii) and (iii) are variants of the proof of [28, Proposition 3.2], which itself follows closely the proof of [5, Theorem 3] extended to the inelastic case in Ref. 9. The starting point is the following differential equation on the moments

$$\frac{d}{dt} m_p = \int_{\mathbb{R}^N} Q(g, g)\,|v|^{2p}\,dv + p\,m_p \quad \text{with} \quad m_p := \int_{\mathbb{R}^N} g\,|v|^{2p}\,dv.$$

Using the same notation as in [28, Proof of Proposition 3.2], we introduce the new rescaled moment function

$$z_p := \frac{m_p}{\Gamma(a\,p + 1/2)}, \qquad Z_p := \max_{k=1,\dots,k_p} \{z_{k+1/2}\, z_{p-k},\ z_k\, z_{p-k+1/2}\},$$

for some fixed $a \geq 2$, and we obtain the differential inequality

$$\frac{dz_p}{dt} \leq A'\, p^{a/2-1/2}\, Z_p - A''\, p^{a/2}\, z_p^{1+1/2p} + p\, z_p \tag{3.2}$$

for any $p = 3/2, 2, \dots$ and for some constants $A', A'' > 0$. Note that (3.2) is nothing but [28, Eq. (3.18)], with an additional term $p\, z_p$ due to the additional term $-\nabla_v \cdot (v\, g)$ in Eq. (1.17).

On the one hand, we remark, by an induction argument, that taking $p_0 = p_0(a, A', A'')$ and $x_0 = x_0(a, A', A'')$ large enough, the sequence of functions $z_p := x^p$ is a sequence of supersolution of (3.2) for any $x \geq x_0$ and $p \geq p_0$. Let us emphasize here that we have to take $a \geq 2$ (*i.e.*, $\eta \leq 1$ in [28, Proof of Proposition 3.2]) because of the additional term $p\, z_p$. On the other hand, choosing x_1 large enough, which may depend on p_0, we have from (i) that the sequence of functions $z_p := x^p$ is a sequence of supersolution of (3.2) for any $x \geq x_1$ and for $p \in \{0, 1/2, ..., p_0\}$. As a consequence, we have proved that there exists $x_2 := \max\{x_0, x_1\}$ such that the set

$$\mathcal{C}_x := \left\{ z = (z_p); \quad z_p \leq x^p \ \ \forall\, p \in \frac{1}{2}\,\mathbb{N} \right\} \tag{3.3}$$

is invariant under the flow generated by the Boltzmann equation for any $x \geq x_2$: if $g(t_1) \in \mathcal{C}_x$ then $g(t_2) \in \mathcal{C}_x$ for any $t_2 \geq t_1$. The end of the proof is exactly similar to that of [28, Proof of Proposition 3.2]. □

The integral upper bound in point (iii) of Theorem 1.3 follows from point (iii) of Proposition 3.1.

3.2. Stability in L^1

The stability result [28, Proposition 3.3] translates for (1.17) into:

$$\|g - h\|_{L^1} + e^{-2T}\, \|(g - h)|v|^2\|_{L^1} \leq e^{C(e^{2T}-1)} \times [\|g_{\rm in} - h_{\rm in}\|_{L^1} + \|(g_{\rm in} - h_{\rm in})|v|^2\|_{L^1}]$$

for any solutions g and h in $C(\mathbb{R}_+, L^1_2) \cap L^\infty(\mathbb{R}_+, L^1_3)$ with initial datum $0 \leq g_{\rm in}, h_{\rm in} \in L^1_3$. This shows that, in the Banach space L^1_2, the evolution semi-group S_t of (1.17) satisfies: for any $t \geq 0$, S_t is (strongly) continuous in any L^1_3 bounded subset of L^1_2. However we shall prove a more precise stability result, working directly on the rescaled Eq. (1.17).

Proposition 3.2. *Let* $0 \le g_{\rm in}, h_{\rm in} \in L^1_3$ *and let* g *and* h *be the two solutions of (1.17) (in* $C(\mathbb{R}_+, L^1_2) \cap L^\infty(\mathbb{R}_+, L^1_3)$*) with* $e \in [0,1]$. *Then there is* $C_{\rm stab} > 0$ *depending only on* B *and* $\sup_{t\ge 0} \|g+h\|_{L^1_3}$ *such that*

$$\forall\, t \ge 0, \quad \|g_t - h_t\|_{L^1_2} \le \|g_{\rm in} - h_{\rm in}\|_{L^1_2}\, e^{C_{\rm stab} t}.$$

Proof of Proposition 3.2. We multiply the equation satisfied by $(g-h)$ by $\phi(t,v) = \mathrm{sgn}(g(t,v) - h(t,v))\,(1+|v|^2)$. We use on the one hand the same arguments as in [28, Proposition 3.4] to treat

$$I = \int_{\mathbb{R}^N} [Q(g,g) - Q(h,h)]\, \phi(t,v)\, dv,$$

which gives

$$I \le C \left(\int_{\mathbb{R}^N} (g+h)\,(1+|v|^3)\, dv \right) \left(\int_{\mathbb{R}^N} |g-h|\,(1+|v|^2)\, dv \right).$$

On the other hand we use that

$$\begin{aligned} - \int_{\mathbb{R}^N} \nabla_v \cdot (v\,(g-h))\, \phi(t,v)\, dv &= -N \int_{\mathbb{R}^N} |g-h|\,(1+|v|^2)\, dv \\ &\quad + \int_{\mathbb{R}^N} |g-h|\, \nabla_v \cdot (v + v|v|^2)\, dv \\ &= 2 \int_{\mathbb{R}^N} |g-h|\, |v|^2\, dv. \end{aligned}$$

This concludes the proof with $C_{\rm stab} = C\, \sup_{t\ge 0} \|g+h\|_{L^1_3} + 2$. □

3.3. Uniform Propagation of Lebesgue Norms

Let us take a normal restitution coefficient $e \in (0,1)$ (the case $e = 0$ can be included in dimension $N = 3$) and $1 < p < +\infty$, and let us consider some initial datum $g_{\rm in} \in L^1_2 \cap L^p$. We compute the time derivative of the L^p norm of the solution g to Eq. (1.17):

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^N} g^p\, dv = \int_{\mathbb{R}^N} Q^+(g,g)\, g^{p-1}\, dv \\ - \int_{\mathbb{R}^N} g^p\, L(g)\, dv - \int_{\mathbb{R}^N} g^{p-1}\, \nabla_v (v\, g)\, dv. \end{aligned}$$

We use the control (1.23), and

$$\int_{\mathbb{R}^N} \nabla_v \cdot (vg)\, g^{p-1}\, dv = N \left(1 - \frac{1}{p}\right) \|g\|^p_{L^p}.$$

Gathering all these estimates, we deduce

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} g^p\,dv \le \int_{\mathbb{R}^N} Q^+(g,g)\,g^{p-1}\,dv - \min\left\{1, N\left(1-\frac{1}{p}\right)\right\}\int_{\mathbb{R}^N} g^p(1+|v|)\,dv.$$

Concerning the gain term, Theorem 2.2 yields, for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} Q^+(g,g)f^{p-1}\,dv \le C_\varepsilon\ \|g\|_{L^1_2}^{1+p\theta}\ \|g\|_{L^p}^{p(1-\theta)} + \varepsilon\ \|g\|_{L^1_2}\,\|g(1+|v|)\|_{L^p}^p.$$

Hence, using the bound C_E on the kinetic energy, if we fix ε such that

$$C_E^{p(1-\theta)}\varepsilon < \frac{1}{2}\min\left\{1, N\left(1-\frac{1}{p}\right)\right\},$$

we obtain

$$\frac{d}{dt}\|g\|_{L^p}^p \le C_+\ \|g\|_{L^p}^{p(1-\theta)} - K_-\ \|g\|_{L^p_{1/p}}^p$$

for some explicit constants $C_+, K_- > 0$. By maximum principle, it shows that the L^p norm of g is uniformly bounded by

$$\sup_{t\ge 0}\|g_t\|_{L^p} \le \max\left\{\left(\frac{C_+}{K_-}\right)^{\frac{1}{p\theta}}, \|g_{\rm in}\|_{L^p}\right\}.$$

The proof for weighted L^p norms is exactly similar. This shows the part concerned with Lebesgue norms in point (i) of Theorem 1.3.

3.4. Non-Concentration in the Rescaled Variables and Haff's Law

In this subsection we give a short proof of Haff's law, even if a stronger pontwise estimate from below on the tail in rescaled variables will be proved in the next section. Let us take a normal restitution coefficient $e \in (0,1)$ (the case $e = 0$ can be included in dimension $N = 3$). Let $f_{\rm in} = g_{\rm in}$ be an initial datum in $L^1_2 \cap L^p$ (with $1 < p < +\infty$). Hence according to the previous subsection, the rescaled solution g to (1.17) with initial datum $g_{\rm in}$ satisfies

$$\sup_{t\ge 0}\|g_t\|_{L^p} \le C_p$$

for some explicit constant $C_p > 0$ depending on the collision kernel and the mass, kinetic energy and L^p norm of $f_{\rm in}$. By using Cauchy-Schwarz inequality, this non-concentration estimate implies that for any $r > 0$

$$\forall\, t \ge 0, \quad \int_{|v|\le r} g(t,v)\,dv \le C\,r^{\frac{p-1}{p}N}.$$

Thus there is $r_0 > 0$ such that

$$\forall\, t \geq 0, \quad \int_{|v| \leq r_0} g(t, v)\, dv \leq \frac{\rho}{2}$$

and thus

$$\begin{aligned} \forall\, t \geq 0, \quad \int_{\mathbb{R}^N} g(t, v)\, |v|^2\, dv &\geq \int_{|v| \geq r_0} g(t, v)\, |v|^2\, dv \\ &\geq r_0^2 \int_{|v| \geq r_0} g(t, v)\, dv \\ &\geq r_0^2 \left(1 - \int_{|v| \leq r_0} g(t, v)\, dv\right) \geq \frac{\rho\, r_0^2}{2}. \end{aligned} \tag{3.4}$$

As a conclusion, gathering (3.1) and (3.4), we have proved that for some constants $C_0, C_1 \in (0, \infty)$ there holds

$$C_0 \leq \mathcal{E}(g(t, \cdot)) \leq C_1,$$

and Haff's law (1.21) follows thanks to (1.20), which proves Theorem 1.2.

Remark 3.3. *The inequality $\mathcal{E}(f(t, \cdot)) \leq M\,(1 + t)^{-2}$ (or equivalently $\mathcal{E}(g(t, \cdot)) \leq C_1$) was already known: see for instance [4, Eqs. (2.5)–(2.6)] where it is proved for a quasi-elastic one-dimensional model with the same evolution Eq. (1.9) on the kinetic energy, by comparison to a differential equation. Indeed the harder part in Haff's law is the first inequality, which means that the solution does not cool down faster than the self-similar profile. As emphasized by the proof above, this is related to the impossiblity of asymptotic concentration in the rescaled Eq. (1.17).*

3.5. Uniform Propagation of Sobolev Norms

Let us take a normal restitution coefficient $e \in (0, 1)$ (the case $e = 0$ can be included in dimension $N = 3$). The study of propagation of regularity and exponential decay of singularities is based on a Duhamel representation of the solution we shall introduce. Let us denote

$$L(t, v) = \left(\int_{\mathbb{R}^N} g(v_*)\, |v - v_*|\, dv_*\right),$$

and

$$S_t g = g(e^{-t} v)\, \exp\left[-Nt - \int_0^t L(s, e^{-(t-s)} v)\, ds\right]$$

the evolution semi-group associated to

$$Tg = -\left(\int_{\mathbb{R}^N} g(v_*)\,|v - v_*|\,dv_*\right) g(v) - \nabla_v \cdot (v\,g).$$

Then the solution of (1.17) represents as

$$g_t = S_t g_{\text{in}} + \int_0^t S_{t-s}\, Q^+(g_s, g_s)\, ds.$$

We give a proposition similar to [32, Proposition 5.2]:

Proposition 3.4. *There are some constants $\alpha > 0$, $\delta > 0$, $K > 0$ and $k > 0$ such that for any $s, \eta \geq 0$, we have*

$$\|S_t g_{\text{in}}\|_{H^{s+\alpha}_\eta} \leq C_{\text{Duh}}\, e^{-Kt}\, \|g_{\text{in}}\|_{H^{s+\alpha}_{\eta+\delta}} \sup_{0\leq \bar{t}\leq t} \|g(\bar{t},\cdot)\|^{s+k}_{H^s_{\eta+\delta}}$$

$$\left\| \int_0^t S_{t-s}\, Q^+(g_s, g_s)\, ds \right\|_{H^{s+\alpha}_\eta} \leq C_{\text{Duh}} \sup_{0\leq \bar{t}\leq t} \|g(\bar{t},\cdot)\|^{s+k}_{H^s_{\eta+\delta}}.$$

Proof of Proposition 3.4. The proof is exactly similar to [32, Proof of Proposition 5.2]. Indeed the semi-group in [32, Proof of Proposition 5.2] is

$$\bar{S}_t g = g(v)\, \exp\left[-\int_0^t L(s, v)\, ds\right]$$

and thus the estimates on the Sobolev norm in v can only improve for S_t according to $\bar{S}_t$. The main tool of [32, Proof of Proposition 5.2], *i.e.*, the Bouchut-Desvillettes-Lu regularity result on Q^+, has been proved in our case in Theorem 2.5. □

Now results follow as in Ref. 32:

Theorem 3.5. *Let $0 \leq g_{\text{in}} \in L^1_2$ be an initial datum and let g be the unique solution of (1.17) in $C(\mathbb{R}_+, L^1_2) \cap L^1(\mathbb{R}_+, L^1_3)$ associated with g_{in}. Then for all $s > 0$ and $\eta \geq 1$, there exists $w(s) > 0$ (explicitly $w(s) = \delta\lceil s/\alpha\rceil$, where α is defined in Proposition 3.4) such that*

$$g_{\text{in}} \in H^s_{\eta+w} \Longrightarrow \sup_{t\geq 0} \|g(t,\cdot)\|_{H^s_\eta} < +\infty$$

with uniform bounds.

Proof of Theorem 3.5. Let $n \in \mathbb{N}$ be such that $n\alpha \geq s$ ($n = \lceil s/\alpha\rceil$). Let $w(s) = \delta\lceil s/\alpha\rceil$. The proof is made by an induction comprising n steps, proving successively that g is uniformly bounded in $H^{i\alpha}_{\eta+\frac{n-i}{n}w}$ for $i = 0, 1, \ldots, n$.

Let us write the induction. The initialisation for $i = 0$, *i.e.*, g uniformly bounded in $L^2_{\eta+w}$ is proved by the previous study of uniform propagation of weighted L^p norms in Sec. 3.3. Now let $0 < i \le n$ and suppose the induction assumption to be satisfied for all $0 \le j < i$. Then proposition 3.4 implies

$$\|S_t g_{\text{in}}\|_{H^{i\alpha}_{\eta+\frac{n-i}{n}w}} \le C_{\text{Duh}}\, e^{-Kt}\, \|g_{\text{in}}(\cdot)\|_{H^{i\alpha}_{\eta+\frac{n-i}{n}w+\delta}} \sup_{0\le t_0\le t} \|g(t_0,\cdot)\|^{i\alpha+k}_{H^{(i-1)\alpha}_{\eta+\frac{n-i}{n}w+\delta}},$$

and

$$\left\| \int_0^t S_{t-s} Q^+(g_s, g_s)\, ds \right\|_{H^{i\alpha}_{\eta+\frac{n-i}{n}w}} \le C_{\text{Duh}} \sup_{0\le t_0\le t} \|g(t_0,\cdot)\|^{i\alpha+k}_{H^{(i-1)\alpha}_{\eta+\frac{n-i}{n}w+\delta}}.$$

Moreover as $i \ge 1$,

$$\eta + \frac{n-i}{n} w + \delta \le \eta + \frac{n-(i-1)}{n} w.$$

Thus, using the induction assumption for $i-1$, g is uniformly bounded in $H^{i\alpha}_{\eta+\frac{n-i}{n}w}$, which concludes the proof. □

3.6. Exponential Decay of Singularities

Let us take a normal restitution coefficient $e \in (0,1)$ (the case $e = 0$ can be included in dimension $N = 3$). In this part we shall follow a similar strategy as in Ref. 32 in order to show that singularities decrease exponentially fast along the flow in rescaled variables. Namely we prove the

Theorem 3.6. *Let* $0 \le g_{\text{in}} \in L^1_2 \cap L^p$, $p \in (1,\infty)$, *and let g be the unique solution of (1.17) in* $C(\mathbb{R}_+, L^1_2) \cap L^1_{\text{loc}}(\mathbb{R}_+, L^1_3)$ *associated with* g_{in}. *Let* $s \ge 0$, $q \ge 0$ *be arbitrarily large. Then g can be written* $g^S + g^R$ *in such a way that*

$$\begin{cases} \sup_{t\ge 0} \left\|g^S_t\right\|_{H^s_q \cap L^1_2} < +\infty, \quad g^S \ge 0 \\ \exists\, \lambda > 0; \ \left\|g^R_t\right\|_{L^1_2} = O\left(e^{-\lambda t}\right). \end{cases}$$

All the constants in this theorem can be computed in terms of the collision kernel, the mass and kinetic energy and L^2 *norm of* g_{in}.

Proof of Theorem 3.6. Assume first that $0 \le g_{\text{in}} \in L^1_2 \cap L^p$, $p \in [2,\infty)$. Then $g_{\text{in}} \in L^2$ and the proof of Theorem 3.6 is exactly similar to [32, Proof of Theorem 5.5] since the only tools of the proof are the stability result, the estimate on the Duhamel representation and the uniform propagation of Sobolev norms, which have been proved respectively in Propositions 3.2, 3.4 and 3.5. The propagation and appearance of moments in L^1 (used in this proof) were proved in

Proposition 3.1. Moreover as was already pointed out in [32, Sec. 7, Remark 3], it is possible with the same arguments to relax the assumptions on the initial datum to $0 \le g_{\rm in} \in L^1_2 \cap L^p$ for any $p \in (1, 2]$ by using the gain of integrability of the gain part of the collision operator. □

Point (i) of Theorem 1.3 is deduced from this theorem.

Remark 3.7. *We do not know how to carry the argument in [32, Theorem 7.2] in order to reduce the assumptions to only* $0 \le g_{\rm in} \in L^1_2$. *The estimates of Abrahamsson on the iterated gain term can be easily extended to the inelastic framework, but the decomposition of Abrahamsson (quoted in the elastic case in [32, Lemma 7.1]) between a part with finite* L^p *norm for some* $p \in (1, 3)$ *and a part which decreases exponentially fast requires a lower bound on the energy. Here for the rescaled inelastic problem we deduce this lower bound from the propagation of* L^p *bounds, which therefore seem compulsory in our method.*

Remark 3.8. *A suggested by this study, the self-similar variables are not only useful for proving the existence of self-similar profiles, but it seems that they also provide the good framework for studying precisely the regularity of the solution. For instance, coming back to the original variables, Theorem 3.6 shows the algebraic decay of singularities for the solutions of (1.1).*

4. SELF-SIMILAR SOLUTIONS AND TAIL BEHAVIOR

In this section we achieve the proofs of Theorems 1.1 and 1.3 by showing the existence of self-similar solutions, and obtaining estimates on their tail and the tail of generic solutions. We consider a normal restitution coefficient $e \in (0, 1)$ (and as before the case $e = 0$ can be included in dimension $N = 3$).

4.1. Existence of Self-Similar Solutions

The starting point is the following result, see for instance [19, Theorem 5.2] or Refs. 3, 18.

Theorem 4.1. *Let* $\mathcal{Y}$ *be a Banach space and* $(S_t)_{t \ge 0}$ *be a continuous semi-group on* $\mathcal{Y}$. *Assume that there exists* $\mathcal{K}$ *a nonempty convex and weakly (sequentially) compact subset of* $\mathcal{Y}$ *which is invariant under the action of* S_t *(that is* $S_t y \in \mathcal{K}$ *for any* $y \in \mathcal{K}$ *and* $t \ge 0$*), and such that* S_t *is weakly (sequentially) continuous on* $\mathcal{K}$ *for any* $t > 0$. *Then there exists* $y_0 \in \mathcal{K}$ *which is stationary under the action of* S_t *(that is* $S_t y_0 = y_0$ *for any* $t \ge 0$*).*

Proof of Theorem 1.1 (existence part). The existence of self-similar solutions follows from the application of this result to the evolution semi-group of (1.17). The continuity properties of the semi-group are proved by the study of the Cauchy problem, recalled in Sec. 3. On the Banach space $\mathcal{Y} = L^1_2$, thanks to the uniform bounds on the L^1_3 and L^p norms, the nonempty convex subset of $\mathcal{Y}$

$$\mathcal{K} = \left\{ 0 \le f \in \mathcal{Y}, \quad \int_{\mathbb{R}^N} f \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix} \quad \text{and} \quad \|f\|_{L^1_3} + \|f\|_{L^p} \le M \right\}$$

is stable by the semi-group provided M is big enough. This set is weakly compact in $\mathcal{Y}$ by Dunford-Pettis Theorem, and the continuity of S_t for all $t \ge 0$ on $\mathcal{K}$ follows from Proposition 3.2. This shows that there exists a nonnegative stationary solution to (1.17) in $L^1_3 \cap L^p$ for any given mass, that is a self-similar solution for the original problem (1.1).

Then one can apply Theorem 3.6, which proves that the stationary solution of (1.17) obtained above belongs to C^∞ (in fact it proves that it belongs to the Schwartz space of C^∞ functions decreasing faster than any polynomial at infinity). Moreover, since the property of being radially symmetric is stable along the flow of (1.17), this sationary solution can be shown to exist within the set of radially symmetric functions by the same arguments. □

4.2. Tail of the Self-Similar Profiles

In this subsection we prove pointwise bounds on the tail behavior of the self-similar solutions. The starting point is the following result extracted from [9, Theorem 1]; notice that it is also a consequence of the construction of invariant sets $\mathcal{C}_x$ for z_p with $a = 2$, as defined in (3.3).

Theorem 4.2. (Bobylev-Gamba-Panferov) *Let G be a steady state of (1.17) with finite moments of all orders. Then G has exponential tail of order* 1, *that is*

$$r^* = \sup \left\{ r \ge 0, \quad \int_{\mathbb{R}^N} G(v) \exp(r|v|)\, dv < +\infty \right\}$$

belongs to $(0, +\infty)$.

Note that if one defines more generally (for $s > 0$)

$$r^*_s = \sup \left\{ r \ge 0, \quad \int_{\mathbb{R}^N} G(v) \exp(r|v|^s)\, dv < +\infty \right\},$$

a simple consequence of this result is that $r^*_s = +\infty$ for any $s < 1$, and $r^*_s = 0$ for any $s > 1$.

First let us prove the pointwise bound from above on the steady state. Since the evolution Eq. (1.17) makes all the moments appear (see Proposition 3.1), we assume that G has finite moments of all orders. Moreover, as discussed above, we can also assume that G is smooth and radially symmetric. We denote $r = |v|$. We thus have the

Proposition 4.3. *Let $G \in C^1$ be a radially symmetric nonnegative steady state of (1.17) with finite moments of all order. Then there exists* $A_1, A_2 > 0$ *such that*

$$\forall\, v \in \mathbb{R}^N, \quad G(v) \le A_1\, e^{-A_2\,|v|}.$$

Proof of Proposition 4.3. The differential equation satisfied by $G = G(r)$ writes

$$Q(G,G) - N\,G - r\,G' = 0.$$

Since G is smooth and integrable, it goes to 0 at infinity. By integrating this equation between $r = R$ and $r = +\infty$, we obtain

$$G(R) = N \int_R^{+\infty} \frac{G(r)}{r}\, dr - \int_R^{+\infty} \frac{Q(G,G)}{r}\, dr.$$

One deduces the following upper bound

$$G(R) \le N \int_R^{+\infty} \frac{G(r)}{r}\, dr + \int_R^{+\infty} \frac{Q^-(G,G)}{r}\, dr.$$

Since $Q^-(G,G) = G\,(G * \Phi)$, we have

$$Q^-(G,G)(v) \le C\,(1 + |v|)\,G.$$

Hence, taking $R \ge 1$ leads to

$$G(R) \le C \int_R^{+\infty} G(r)\, r^{N-1}\, dr.$$

Finally, since we have by Theorem 4.2

$$\int_0^{+\infty} G(r)\, \exp(A_2\, r)\, r^{N-1}\, dr \le A_0 < +\infty$$

for some constants $A_0, A_2 > 0$, we deduce that

$$G(R) \le C \int_R^{+\infty} G(r)\, r^{N-1}\, dr \le C\, A_0\, \exp(-A_2\, R) = A_1\, \exp(-A_2\, R).$$

This concludes the proof. □

For the pointwise lower bound, we give here a proof based on a maximum principle argument, inspired from the works.[19,20] We shall in the next subsection give a more general result for generic solutions of (1.17), based on the spreading effect of the gain term and the dispersion (or transport) effect of the evolution semi-group of (1.17) (due to the anti-drift term) in the spirit of Refs. 13, 33.

Proposition 4.4. *Let $G \in C^1$ be a nonnegative steady state of (1.17) with finite moments of orders* 0 *and* 2 *and which is not identically equal to* 0*. Then there exists $a_1, a_2 > 0$ such that*

$$\forall\, v \in \mathbb{R}^N, \quad G(v) \geq a_1\, e^{-a_2\,|v|}.$$

We first start with a lemma.

Lemma 4.5. *For any $r_0, a_1, \rho_0, \rho_1 > 0$, there exists $a_2 > 0$ such that the function $h(v) := a_1\ \exp(-a_2\,|v|)$ satisfies*

$$\forall\, v,\ |v| \geq r_0, \quad Q^-(g,h) + \nabla_v \cdot (v\,h) \leq 0 \tag{4.1}$$

for any function g such that

$$\int_{\mathbb{R}^N} g(v)\,dv = \rho_0, \quad \int_{\mathbb{R}^N} g(v)\,|v|\,dv = \rho_1.$$

Proof of Lemma 4.5. On the one hand, it is straightforward that

$$Q^-(g,h) := (g * \Phi)\,h \leq (\rho_1 + \rho_0\,|v|)\,h.$$

On the other hand, simple computations show that

$$\nabla_v \cdot (v\,h) = (N - a_2\,|v|)\,h.$$

Gathering these two inequalities there holds

$$\forall\, v,\ |v| \geq r_0, \quad Q^-(g,h) + \nabla_v \cdot (v\,h) \leq (\rho_1 + N + \rho_0\,|v| - a_2\,|v|)\,h \leq 0$$

for a_2 large enough. □

Proof of Proposition 4.4. Since $G \in C^1$ and it is radially symmetric, there holds $G'(0) = 0$. As a consequence, the equation satisfied by G reads in $v = 0$

$$Q(G,G)(0) - N\,G(0) = 0$$

and then

$$G(0) = \frac{Q^+(G,G)(0)}{N} > 0$$

since G is not zero everywhere. By continuity, $G(v) > 2\,a_1$ on $B(0, r_0)$ for some $a_1, r_0 > 0$.

Let us define

$$\rho_0 := \int_{\mathbb{R}^N} G(v)\,dv, \quad \rho_1 := \int_{\mathbb{R}^N} G(v)\,|v|\,dv,$$

and $a_2 > 0$ associated to r_0, a_1, ρ_0, ρ_1 by Lemma 4.5. On the one hand $h(v) := a_1 \exp(-a_2 |v|)$ satisfies (4.1) for $g = G$ and, on the other hand, G satisfies

$$\forall\, v \in \mathbb{R}^N, \quad Q^-(G, G) + \nabla_v(v\, G) = Q^+(G, G) \geq 0. \tag{4.2}$$

Introducing the auxiliary function $W := G - h$, we deduce from (4.1) and (4.2)

$$\forall\, v, \;\; |v| \geq r_0, \quad (G * \Phi)\, W + \nabla_v(v\, W) \geq 0$$

and $W(r_0) = G(r_0) - h(r_0) \geq G(r_0)/2 > 0$. By the Gronwall Lemma (using that all the functions involved in this inequality are radially symmetric), we get $W(v) \geq 0$ for any v, $|v| \geq r_0$, which concludes the proof. □

4.3. Positivity of the Rescaled Solution

We start with three technical lemmas.

Lemma 4.6. *Let g_0 satisfies for $p \in (1, \infty)$*

$$\int_{\mathbb{R}^N} g_0\, dv = 1, \quad \int_{\mathbb{R}^N} g_0\, |v|^2\, dv \leq C_1, \quad \int_{\mathbb{R}^N} g_0^p\, dv \leq C_2. \tag{4.3}$$

There exist $R > r > 0$ and $\eta > 0$ depending only on C_1, C_2, and $(v_i)_{i=1,\ldots,4}$ such that $|v_i| \leq R$, $i = 1, \ldots, 4$, and $|v_i - v_j| \geq 3r$ for $1 \leq i \neq j \leq 3$, and

$$\int_{B(v_i, r)} g_0(v)\, dv \geq \eta \quad \textit{for} \quad i = 1,\, 2,\, 3, \tag{4.4}$$

$$\forall\, w_i \in B(v_i, r), \quad E^e_{w_3, w_4} \cap S^e_{w_1, w_2} \; \textit{is a sphere of radius larger than } r, \tag{4.5}$$

where $E^e_{v,'v}$ stands for the plane defined in Proposition 1.5 and S^e_{v,v_*} stands for the sphere of all possibles post-collisional velocity v' defined by (1.26).

Proof of Theorem 4.6. Let C_R denotes the hypercube $[-R, R]^N$ centered at $v = 0$ with length $2R > 0$. Thanks to the mass condition and the energy bound in (4.3), for R large enough, there holds

$$\int_{C_R} g_0\, dv \geq \frac{1}{2}. \tag{4.6}$$

Then we define $(K_i)_{i=1,\ldots,I}$ the familly of $I = (2\, R/r)^N$ hypercubes of length $r > 0$ (with $R/r \in \mathbb{N}$), included in C_R and such that the union of K_i is almost equal to C_R. For any given $\lambda > 0$ to be later fixed, we may find $r > 0$ such that

$$\begin{aligned} \int_{K_i + B(0,\lambda r)} g_0\, dv &\leq |K_i + B(0, \lambda r)|^{1/p'} \left(\int_{K_i + B(0,\lambda r)} g_0^p\, dv \right)^{1/p} \\ &\leq C\, [(\lambda + 1) r]^{N/p'} \leq 1/4 \end{aligned} \tag{4.7}$$

for any $i = 1, \dots, I$. Hence we can choose K_{i_0} such that the mass of g_0 in K_i is maximal for $i = i_0$. Because of (4.6) there holds

$$\int_{K_{i_0}} g_0 \, dv \geq 1/4 \, (2\, R/r)^{-N}. \tag{4.8}$$

Gathering (4.6) and (4.7) we may find $K_{j_0} \subset C_R$ such that $\mathrm{dist}(K_{i_0}, K_{j_0}) > \lambda r$ and (4.8) also holds for $i = j_0$.

Next, we fix $\lambda := 200\,\beta$. We define v_1 (respectively v_2) as the center of the hypercube K_{i_0} (respectively K_{j_0}), and $v_3 = (v_1 + v_2)/2$ and $v_4 = v_2$. Then we have

$$\Omega(v_3, v_4) = v_1 + \frac{\beta^{-1}}{2} (v_2 - v_1) \in [v_1, v_2],$$

which implies

$$|\Omega - v_1| = \frac{\beta^{-1}}{2} |v_2 - v_1| \geq \frac{\beta^{-1}}{2} (\lambda\, r) \geq 100\, r.$$

Thus $E^e_{v_3, v_4} \cap S^e(v_1, v_2)$ is a $(N-2)$-dimensional sphere of radius larger than $100\, r$ (because $B(\Omega, 100r)$ is included in the convex hull of $S^e(v_1, v_2)$), and (4.5) follows straightforwardly. □

Lemma 4.7. *Let us fix $R > r > 0$ and $\eta > 0$. Then there exists $\delta_0 > 0$, $\eta_0 > 0$, $\xi_0 \in (0, 1)$ (depending on $R > r > 0$, $\eta > 0$ and B) such that, for any functions f, h, ℓ satisfying (4.4)–(4.5) for some velocities $(v_i)_{i=1,\dots,4}$ such that $|v_i| \leq R$, $i = 1, \dots, 4$ and $|v_i - v_j| \geq 3r$, $1 \leq i \neq j \leq 3$, and for any $\xi \in (\xi_0, 1)$, there holds*

$$Q^+(f, Q^+_\xi(h, \ell)) \geq \eta_0 \, \mathbf{1}_{B(v_3, \delta_0)},$$

where we define here and below $Q^+_\xi(\cdot, \cdot)(v) = Q^+(\cdot, \cdot)(\xi\, v)$.

Proof of Theorem 4.7. We first establish a convenient formula to handle representations of the iterated gain term. For any f, h and ℓ and any $v \in \mathbb{R}^N$ there holds (setting $'v = w$ and $'v_* = w_*$)

$$Q^+(f, Q^+_\xi(h, \ell))(v) = C'_b \int_{\mathbb{R}^N} \frac{f(w)}{|v - w|} \left\{ \int_{E^e_{v,w}} Q^+_\xi(h, \ell)(w_*) \, dw_* \right\} dw.$$

From the following identity

$$Q^+_\xi(h, \ell)(w_*) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h(w_1)\, \ell(w_2)\, Q^+_\xi(\delta_1, \delta_2)(w_*) \, dw_1 dw_2$$

where δ_i stands for the Dirac measure at w_j, the term between brackets, that we denote by A, write

$$A(v,w) = \int_{\mathbb{R}^N\times\mathbb{R}^N} h(w_1)\,\ell(w_2) \times \left\{ \lim_{\varepsilon\to 0} \frac{1}{2\,\varepsilon} \int_{\mathbb{R}^N} Q^+(\delta_1,\delta_2)(\xi\, w_*)\, \Xi_\varepsilon(w_*)\, dw_* \right\} dw_1\, dw_2$$

where Ξ_ε denotes the indicator function of the set $\{w_*;\ \text{dist}(w_*, E_{v,w}) < \varepsilon\}$. Denoting now by D_ε the integral just after the limit sign in the term between brackets, and using the weak formulation (1.25), there holds

$$\begin{aligned} D_\varepsilon &= \frac{\xi^{-1}}{2\,\varepsilon} \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} \delta_1(z)\,\delta_1(z_*)\,|z-z_*|\, b(\sigma\cdot\hat{z})\, \Xi_\varepsilon(\xi^{-1}\, z')\, d\sigma\, dz\, dz_* \\ &= |w_1 - w_2|\, \xi^{-1}\, C_b \int_{\mathbb{S}^{N-1}} \frac{\Xi_\varepsilon(z'\,\xi^{-1})}{2\varepsilon}\, d\sigma, \end{aligned}$$

where in these integrales z' is defined from (z, z_*, σ) and next from (w_1, w_2, σ) thanks to formula (1.26). We define $\xi_0 = (1 + r/R)^{-1}$ in such a way that $|\xi^{-1}\, z' - z'| \le r$ for any $z' \in B(0,R)$ and $\xi \in (\xi_0, 1)$. Taking $v \in B(v_3, r)$, $w \in B(v_4, r)$, $w_1 \in B(v_1, r)$, $w_2 \in B(v_2, r)$, we have thanks to (4.5) and $|w_2 - w_1| \ge r$:

$$D_0(v, w, w_1, w_2) := \lim_{\varepsilon\to 0} D_\varepsilon \ge r\, \xi_0^{-1}\, C_b\, C\, r^{N-2}.$$

As a consequence, for any $v \in B(v_3, r)$,

$$\begin{aligned} Q^+(f, Q^+_\xi(h,\ell))(v) &\ge Q^+(f\,\mathbf{1}_{B(v_4,r)}, Q^+(h\,\mathbf{1}_{B(v_1,r)}, \ell\,\mathbf{1}_{B(v_2,r)}))(v) \\ &\ge C_b' \int_{B(v_1,r)}\int_{B(v_2,r)}\int_{B(v_4,r)} \frac{f(w)}{|v-w|}\, h(w_1)\,\ell(w_2)\, D_0\, dw_1\, dw_2\, dw \\ &\ge C_b'\, \eta^3\, \frac{1}{2\,R}\, r\, \xi_0^{-1}\, C_b\, C\, r^{N-2} =: \eta_0. \end{aligned}$$

This concludes the proof. □

Lemma 4.8. *For any $\bar{v} \in \mathbb{R}^N$ and $\delta > 0$, there exists $\kappa = \kappa(\delta) > 0$ such that*

$$\mathcal{Q}^+(v) := Q^+\big(\mathbf{1}_{B(\bar{v},\,\delta)}, \mathbf{1}_{B(\bar{v},\,\delta)}\big) \ge \kappa\, \mathbf{1}_{B\left(\bar{v},\,\frac{\sqrt{5}}{2}\,\delta\right)}. \tag{4.9}$$

Proof of Lemma 4.8. The homogeneity property (1.13) of Q^+ and the invariance by translation allow to reduce the proof of (4.9) to the case $\bar{v} = 0$ and $\delta = 1$. The invariance by rotations implies that $\mathcal{Q}^+$ is radially symmetric and the homogeneity property again allows to conclude that the support of $\mathcal{Q}^+$ is a ball B'. More precisely, taking a C^∞ radially symmetric function ϕ such that $\phi > 0$ on $B = B(0,1)$ and $\phi \le \mathbf{1}_B$ on $\mathbb{R}^N$, we have $Q^+(\phi,\phi)$ is continuous, $\mathcal{Q}^+ \ge Q^+(\phi,\phi)$ on $\mathbb{R}^N$ and $Q^+(\phi,\phi) > 0$ on the ball B'. As a consequence, for any ball B'' strictly included in B', there exists $\kappa > 0$ such that $\mathcal{Q}^+ \ge \kappa\, \mathbf{1}_{B''}$. In order to conclude, we just need to estimate the support of $\mathcal{Q}^+$.

Let us fix $R \in (0,1)$ and choose $'v, 'v_* \in B(0,1)$ such that $'v \perp 'v_*$, $|'v| = |'v_*| = R$. Then for any $\sigma \in \mathbb{S}^{N-1}$, $\sigma \perp 'v - 'v_*$, the function $\mathcal{Q}^+$ is positive at the post-collisional associated velocity v defined by

$$v = \frac{'v + 'v_*}{2} + \frac{1-e}{4}\,('v - 'v_*) + \frac{1+e}{4}\,|'v - 'v_*|\,\sigma.$$

Remarking that $|'v + 'v_*|^2 = |'v - 'v_*|^2 = 2\,R^2$, $('v - 'v_*)\cdot('v + 'v_*) = 0$ and $('v + 'v_*)\cdot\sigma = \sqrt{2}\,R$, we easily compute

$$|v|^2 = R^2 \left[1 + \left(\frac{1+e}{2}\right)^2\right] > \frac{5}{4}\,R^2,$$

and the radius of B' is strictly larger than $\sqrt{5}/2$. □

Theorem 4.9. *Let $g_{\rm in}$ satisfy the hypothesis of Theorem 1.3 and let g be the solution to the rescaled Eq. (1.17) associated to the initial datum $g_{\rm in}$. Then for any $t_* > 0$, $g(t,\cdot) > 0$ a.e. on $\mathbb{R}^N$ for any $t \ge t_*$, and there exists $a_1, a_2, c > 0$ such that*

$$\forall\, t \ge t_*, \quad g(t,v) \ge a_1\, e^{-a_2\,|v|}\, \mathbf{1}_{|v| \le c\, e^{t-t_*}} \text{ for a.e. } v \in \mathbb{R}^N.$$

Proof of Theorem 4.9. We split the proof into four steps.

Step 1. The starting point is the evolution equation satisfied by g written in the form

$$\partial_t g + v\cdot\nabla_v g + (N + |v|)\, g = Q^+(g,g) + (|v| - L(g))\, g.$$

Let us introduce the semi-group S_t associated to the operator $v\cdot\nabla_v + \lambda(v)$, where $\lambda(v) := N + |v|$. Thanks to the Duhamel formula and (1.23), we have

$$g(t,\cdot) \ge S_t\, g(0,\cdot) + \int_0^t S_{t-s}\, Q^+(g(s,\cdot), g(s,\cdot))\, ds, \tag{4.10}$$

where the semi-group S_t is defined by

$$(S_t\, h)(v) = h(v\, e^{-t})\, \exp\left(-\int_0^t \lambda(v\, e^{-s})\, ds\right).$$

Notice that

$$\left(- \int_0^t \lambda(v\, e^{-s})\, ds \right) \geq -(|v| + N\, t).$$

Step 2. Let us fix $t_0 > 0$ and define $\tilde{g}_0(t, \cdot) := g(t_0 + t, \cdot)$. Using twice the Duhamel formula (4.10), we find

$$\begin{aligned} \tilde{g}_0(t, \cdot) &\geq \int_0^t S_{t-s}\, Q^+ \left(\tilde{g}_0(s, \cdot), \int_0^s S_{s-s'}\, Q^+(\tilde{g}_0(s', \cdot), \tilde{g}_0(s', \cdot))\, ds' \right) ds \\ &\geq \int_0^t \int_0^s S_{t-s}\, Q^+ \left(S_s \tilde{g}_0, S_{s-s'}\, Q^+(S_{s'} \tilde{g}_0, S_{s'} \tilde{g}_0) \right) ds'\, ds. \end{aligned} \tag{4.11}$$

We apply now Lemma 4.6 to $\tilde{g}_0$ and set $R_0 := 2R$. Since S_t is continuous in L^1, there exists $T_1 > 0$, such that for any $s \in [0, T_1]$, there holds

$$\int_{B(v_i, r)} S_s(\tilde{g}_0)(v)\, dv \geq \eta/2 \quad \text{for} \quad i = 1, 2, 3,$$

and $e^{-T_1} > \xi_0$. For $v \in B(0, R_0)$ and $t \in [0, T_1]$ we may estimate $S_t h$ from below in the following way

$$(S_t\, h)(v) \geq \gamma\, h_{e^{-t}}(v)$$

for some constant $\gamma = \gamma_{R_0, T_1}$. The bound from below (4.11) then yields (using Lemma 4.7)

$$\begin{aligned} \tilde{g}_0(t, \cdot) &\geq \gamma^2 \int_0^t \int_0^s Q^+_{e^{s-t}}(S_s \tilde{g}_0, Q^+_{e^{s'-s}}(S_{s'} \tilde{g}_0, S_{s'} \tilde{g}_0))\, ds'\, ds \\ &\geq \gamma^2 \int_0^t \int_0^s \eta_0\, \mathbf{1}_{v\, e^{s-t} \in B(v_3, r)}\, ds'\, ds. \end{aligned}$$

We have then proved that there exists $T_1 > 0$ and for any $t_1 \in (0, T_1/2]$ there exists $\eta_1 > 0$ such that (for some $\bar{v} \in B(0, R)$)

$$\forall\, t \in [0, T_1/2], \quad \tilde{g}_1(t, \cdot) := \tilde{g}_0(t + t_1, \cdot) \geq \eta_1\, \mathbf{1}_{B(\bar{v}, \delta_1)}.$$

Step 3. Using again the Duhamel formula (4.10) and the preceding step we have

$$\tilde{g}_1(t, \cdot) \geq \int_0^t S_{t-s}\, Q^+(\tilde{g}_1(s, \cdot), \tilde{g}_1(s, \cdot))\, ds.$$

Thanks to Lemma 4.7, on the ball $B(0, R_0)$, there holds

$$\tilde{g}_1(t, \cdot) \geq \eta_1^2 \int_0^t S_{t-s}\, Q^+(\mathbf{1}_{B(\bar{v}, \delta_1)}, \mathbf{1}_{B(\bar{v}, \delta_1)})\, ds$$

$$\geq \eta_1^2\, \kappa(\delta_1)\, e^{-(R_0+N\,t)} \int_0^t \mathbf{1}_{e^{-t}\, v \in B(\bar{v},\sqrt{5}\,\delta_1/2)}\, ds$$

$$\geq \eta_1^2\, \kappa(\delta_1)\, e^{-(R_0+N\,T_1)}\, t\, \mathbf{1}_{B(\bar{v},\sqrt{19}\,\delta_1/4)}$$

on $[0, T_2]$ with $T_2 \in (0, T_1/2]$ small enough, and then

$$\tilde{g}_2(t,\cdot) := \tilde{g}_1(t+t_2,\cdot) \geq \eta_2\, \mathbf{1}_{B(\bar{v},\delta_2)} \quad \text{on} \quad [0, T_2/2]$$

with $\delta_2 := \sqrt{19}\,\delta_1/4$ and $t_2 \in (0, T_2/2]$ arbitrarily small, $\eta_2 > 0$. Repeating the argument we obtain

$$\tilde{g}_k(t,\cdot) := g\left(t + \sum_{i=0}^{k} t_i, \cdot\right) \geq \eta_k\, \mathbf{1}_{B(\bar{v},\delta_k)} \text{ on } [0, T_k/2], \quad \text{with} \quad \delta_k := (\sqrt{19}/4)^k\, \delta_1$$

with $k \geq 1$ and some $t_i \in [0, T_i/2]$ arbitrarily small, $\eta_k > 0$. As a consequence, taking k large enough in such a way that $\delta_k R_0$, we get for some explicit constant $\eta_* > 0$ and some (arbitrarily small) time $t_* > 0$

$$\forall\, t_0 \geq 0, \quad g(t_* + t_0, \cdot) \geq \eta_*\, \mathbf{1}_{B(0,R)}. \tag{4.12}$$

Step 4. Coming back to the Duhamel formula (4.10) where we only keep the first term, we have, for any $t_0 \geq 0$,

$$\forall\, t \geq t_*, \quad g(t_0 + t, v) \geq \eta_*\, \mathbf{1}_{|v| \leq R\, e^{t-t_*}}\, \exp(-|v| - N\,(t - t_*)).$$

As a consequence, for any $t > t_*$,

$$g(t,v) \geq \mathbf{1}_{|v| \leq R\, e^{t-t_*}} \left(\sup_{s \in [0, t-t_*]} \mathbf{1}_{|v| = R\, e^s}\, \exp(-|v| - N\,s) \right) \tag{4.13}$$

$$\geq \mathbf{1}_{|v| \leq R\, e^{t-t_*}} \left(\sup_{s \in [0, t-t_*]} \mathbf{1}_{|v| = R\, e^s} \right) \exp(-|v| - N\, \ln^+(|v|/R)),$$

and we conclude gathering (4.12) and (4.13). □

It is straightforward that Theorem 4.9 implies the lower bound in point (iii) of Theorem 1.3.

5. PERSPECTIVES

As a conclusion, we discuss some possible perspectives arising from our study (partial answers to them shall be studied in a forthcoming work[29]).

Let us denote

$$\mathcal{P} = \Big\{ G \in C^\infty,\ G \text{ radially symmetric}, \quad \exists\, a_1, a_2, A_1, A_2 > 0 \ \mid\ a_1 e^{-a_2|v|} \leq G(v) \leq A_1 e^{-A_2|v|} \Big\}.$$

Conjecture 1. For any mass $\rho > 0$, the self-similar profile G_ρ with mass ρ and momentum 0 is unique.

If Conjecture 1 is true, the natural conjecture is

Conjecture 2. (Strong version) For any initial datum with mass ρ and momentum 0 (maybe with some regularity and/or moment assumptions), the associated solution satisfies (in rescaled variables)

$$g_t \to_{t\to\infty} G_\rho,$$

where G_ρ is the steady state of (1.17) with mass ρ and momentum 0.

A relaxed version can be

Conjecture 2. (Weak version) For any initial datum with mass ρ and momentum 0 the associated solution satisfies (in rescaled variables)

$$g_t = g_t^S + g_t^R$$

with $g_t^S \in \mathcal{P}$ and $g_t^R \to_{t\to\infty} 0$ in L^1.

Note that the weak version of Conjecture 2 still makes sense when the self-similar profile with mass ρ and momentum 0 is not unique and even if there is no convergence towards some self-similar profile (which could be the case for instance if the solution in rescaled variables "oscillates" asymptotically between several self-similar profiles).

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